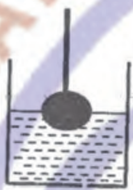
CSIR/UGC/NET MATHEMATICS DECEMBER 2011

PART A

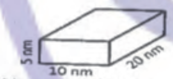
1. There is an equilateral triangle in the XY - plane with its center at the origin. The distance of its sides from the origin is 3.5 cm. The area of its circumference in cm^2 is
   (a) 38.5
   (b) 49
   (c) 63.65
   (d) 154

2. A sphere of iron of radius $R/2$ fixed to one end of a string was lowered into water in a cylindrical container of base radius R to keep exactly half the sphere dipped. The rise in the level of water in the container will be

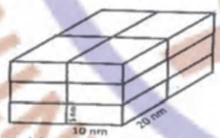


   (a) $R/3$
   (b) $R/4$
   (c) $R/8$
   (d) $R/12$

3. A crystal grows by stacking of unit cells of $10\times20\times5$ nm size as shown in the diagram given below. How many unit cells will make a crystal of 1 cm^3 volume?



Unit Cell (not to scale)

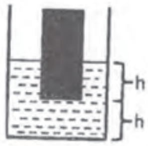


Crystal (not to scale)

   (a) 10^6
   (b) 10^9
   (c) 10^{12}
   (d) 10^{18}

4. What is the value of $\frac{1}{1\times2}+\frac{1}{2\times3}+\frac{1}{3\times4}+\cdots$ to ∞?
   (a) 2/3
   (b) 1
   (c) 2
   (d) ∞

5. A solid cylinder of basal area A was held dipped in water in a cylindrical vessel of basal area $2A$ vertically such that a length h of the cylinder is immersed. The lower tip of the cylinder is at a height h from the base of the vessel. What will be the height of the water in the vessel when the cylinder is taken out?
   (a) $2h$
   (b) $\frac{3}{2}h$



   (c) $\frac{4}{3}$
   (d) $\frac{5}{4}h$

6. Of all the triangles that can be inscribed in a semi-circle of radius R with the diameter as one side, the biggest one has the area
   (a) R^2
   (b) $R^2\sqrt{R}$
   (c) $R^2\sqrt{3}$
   (d) $2R^2$

7. Choose the largest number:
   (a) 2^{500}
   (b) 3^{400}
   (c) 4^{300}
   (d) 5^{200}

8. A daily sheet calender of the year 2013 contains sheets of 10×10 cm size. All the sheets of the calender are spread over the floor of a room of $5m\times7.3m$ size. What percentage of the floor will be covered by these sheets?
   (a) 0.1
   (b) 1
   (c) 10
   (d) 100

9. How many rectangles (which are not squares) are there in the following figure?
   (a) 56
   (b) 70
   (c) 86
   (d) 100

10. Define $a\otimes b=\mathrm{lcm}(a,b)+\gcd(a,b)$ and $a\oplus b=a^b+b^a$. What is the value of $(1\oplus2)\otimes(3\oplus4)$? Here lcm = least common multiple and gcd = greatest common divisor.
   (a) 145
   (b) 286
   (c) 436
   (d) 572

11. A square pyramid is to be made using a wire such that only one strand of wire is used for each edge. What is the minimum number of times that the wire has to be cut in order to make the pyramid?
   (a) 3
   (b) 7
   (c) 2
   (d) 1

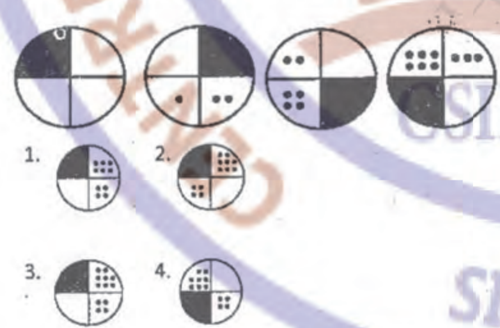
12. A crow is flying along a horizontal circle of radius R at a height R above the horizontal ground. Each of a number of men on the ground found that the angular height of the crow was fixed angle $\theta(< 45^o)$ when it was closest to him. Then all these men must be on a circle on the ground with a radius
   (a) $R + R\sin\theta$
   (b) $R + R\cos\theta$
   (c) $R + R\tan\theta$
   (d) $R + R\cot\theta$

13. How many pairs of positive integers have gcd 20 and lcm 600? (gcd = greatest common divisor; lcm = least common multiple)
   (a) 4
   (b) 0
   (c) 1
   (d) 7

14. During an evening party, when Mr. Black, Ms. Brown and Ms. White met, Ms. Brown remarked, "It is interesting that our dresses are white, black or brown, but for each of us the name does not match the color of the dress!". Ms. White replied, "But your white dress does not suit you!". Pick the correct answer.
   (a) Ms. White's dress was brown.
   (b) Ms. Black's dress was white.
   (c) Ms. White's dress was black.
   (d) Ms. Black's dress was black.

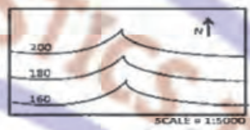
15. Two integers are picked at random from the first 15 positive integers without replacement. What is the probability that the sum of the two numbers is 20?
   (a) 3/4
   (b) 1/21
   (c) 1/105
   (d) 1/20

16. Identify the next figure in the sequence



17. In a customer survey conducted during Monday to Friday, of the customers who asked for child care facilities in super markets, 23% were men and the rest, women. Among them, 19.9% of the women and 8.8% of the men were willing to pay for the facilities.
   A. What is the ratio of the men to women customers who wanted child care facilities?
   B. If the survey had been conducted during the weekend instead, how will the result change?
   With the above data,
   (a) Only A can be answered
   (b) Only B can be answered
   (c) Both A and B can be answered
   (d) Neither A nor can be answered

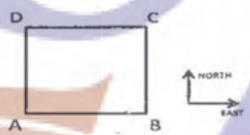
18. The map given below shows contour lines which connect points of equal ground surface elevation in a region. Inverted 'V' shaped portions of contour lines represent a valley along which a river flows. What is the downstream direction of the river?



   (a) North
   (b) South
   (c) East
   (d) west

19. During a summer vacation, of 20 friends from a hostel, each wrote a letter to each of all others. The total number of letters written was
   (a) 20
   (b) 400
   (c) 200
   (d) 380

20. A person has to cross a square field by going from A to C. The person is only allowed to move towards the east or towards the north or use a combination of these movements. The total distance traveled by the person



   (a) depends on the length of each step
   (b) depends on the total number of steps
   (c) is different for different paths
   (d) is the same for all paths

PART B

21. The determinant of the matrix

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 & 2 & 0 \\ 0 & 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 2 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 & 1 & 0 \\ 2 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$ is

(1) 0 **(2)** -9
(3) -27 **(4)** 1

22. Which of the following statements is true?

1. $\lim\limits_{x\to\infty} \frac{\log x}{x^{1/2}} = 0$ and $\lim\limits_{x\to\infty} \frac{\log x}{x} = \infty$
2. $\lim\limits_{x\to\infty} \frac{\log x}{x^{1/2}} = \infty$ and $\lim\limits_{x\to\infty} \frac{\log x}{x} = 0$
3. $\lim\limits_{x\to\infty} \frac{\log x}{x^{1/2}} = 0$ and $\lim\limits_{x\to\infty} \frac{\log x}{x} = 0$
4. $\lim\limits_{x\to\infty} \frac{\log x}{x^{1/2}} = 0$ but $\lim\limits_{x\to\infty} \frac{\log x}{x}$ does not exist

23. For a positive integer n, let P_n denote the space of all polynomials $p(x)$ with coefficients in $\mathbb{R}$ such that $\deg p(x) \le n$, and let B_n denote the standard basis of P_n given by $B_n = \{1, x, x^2, \ldots, x^n\}$. If $T : P_3 \to P_4$ is the linear transformation defined by $T(p(x)) = x^2 p'(x) + \int_0^x p(t)\,dt$ and $A = (a_{ij})$ is the 5×4 matrix of T with respect to standard bases B_3 and B_4, then

1. $a_{32} = \frac{3}{2}$ and $a_{33} = \frac{7}{3}$
2. $a_{32} = \frac{3}{2}$ and $a_{33} = 0$
3. $a_{32} = 0$ and $a_{33} = \frac{7}{3}$
4. $a_{32} = 0$ and $a_{33} = 0$

24. Consider the power series $\sum_{n\ge1} a_n z^n$ where a_n = number of divisors of n^{50}. Then the radius of convergence of $\sum_{n\ge1} a_n z^n$ is

(1) 1 **(2)** 50
(3) $\frac{1}{50}$ **(4)** 0

25. Let p be a prime number. The order of a p-Sylow subgroup of the group $GL_{50}(\mathbb{F}_p)$ of invertible 50×50 matrices with entries from the finite field $\mathbb{F}_p$, equals:

(1) p^{50} **(2)** p^{125}
(3) p^{1250} **(4)** p^{1225}

26. Let X be a connected subset of real numbers. If every element of X is irrational, then the cardinality of X is

(1) infinite **(2)** countably infinite
(3) 2 **(4)** 1

27. Suppose the matrix

$$A = \begin{pmatrix} 40 & -29 & -11 \\ -18 & 30 & -12 \\ 26 & 24 & -50 \end{pmatrix}$$

has a certain complex number $\lambda \ne 0$ as an eigenvalue. Which of the following numbers must also be an eigenvalue of A?

(1) $\lambda + 20$ **(2)** $\lambda - 20$
(3) $20 - \lambda$ **(4)** $-20 - \lambda$

28. Define $f:[0,1]\to[0,1]$ by $f(x)=\frac{2^k-1}{2^k}$ for $x\in\left[\frac{2^{k-1}-1}{2^{k-1}},\frac{2^k-1}{2^k}\right]$, $k\geq 1$.

Then f is a Riemann-integrable function such that

1. $\int_0^1 f(x)dx = \frac{2}{3}$
2. $\frac{1}{2} < \int_0^1 f(x)dx < \frac{2}{3}$
3. $\int_0^1 f(x)dx = 1$
4. $\frac{2}{3} < \int_0^1 f(x)dx < 1$

29. Let $A=\begin{bmatrix}0&0&0&-4\\1&0&0&0\\0&1&0&5\\0&0&1&0\end{bmatrix}$.

Then a Jordan canonical form of A is

(1) $\begin{bmatrix}-1&0&0&0\\0&1&0&0\\0&0&2&0\\0&0&0&-2\end{bmatrix}$ (2) $\begin{bmatrix}-1&1&0&0\\0&1&0&0\\0&0&2&0\\0&0&0&-2\end{bmatrix}$

(3) $\begin{bmatrix}1&1&0&0\\0&1&0&0\\0&0&2&0\\0&0&0&-2\end{bmatrix}$ (4) $\begin{bmatrix}-1&1&0&0\\0&-1&0&0\\0&0&2&0\\0&0&0&-2\end{bmatrix}$

30. Let $I_r=\int_{C_r}\frac{dz}{z(z-1)(z-2)}$, where $C_r=\{z\in\mathbb{C}:|z|=r\}, r>0$. Then

1. $I_r=2\pi i$ if $r\in(2,3)$
2. $I_r=\frac{1}{2}$ if $r\in(0,1)$
3. $I_r=-2\pi i$ if $r\in(1,2)$
4. $I_r=0$ if $r>3$

31. Let A be a 5×4 matrix with real entries such that the space of all solutions of the linear system $AX^t=[1,\ 2,\ 3,\ 4,\ 5]^t$ is given by $\{[1+2s,\ 2+3s,\ 3+4s,\ 4+5s]^t : s\in\mathbb{R}\}$. (Here M^t denotes the transpose of a matrix M.) Then the rank of A is equal to

(1) 4 (2) 3
(3) 2 (4) 1

32. Let A be a 3×3 matrix with real entries such that $\det(A)=6$ and the trace of A is 0. If $\det(A+I)=0$, where I denotes the 3×3 identity matrix, then the eigenvalues of A are

(1) $-1,\ 2,\ 3$ (2) $-1,\ 2,\ -3$
(3) $1,\ 2,\ -3$ (4) $-1,\ -2,\ 3$

33. Let $p(z,w)=\alpha_0(z)+\alpha_1(z)w+\ldots+\alpha_k(z)w^k$, where $k\geq 1$ and $\alpha_0,\ldots,\alpha_k$ are non-constant polynomials in the complex variable z. Then $\{(z,w)\in\mathbb{C}\times\mathbb{C} : p(z,w)=0\}$ is

1. bounded with empty interior
2. unbounded with empty interior
3. bounded with nonempty interior
4. unbounded with nonempty interior

34. Let (X,d) be a metric space and let $A\subseteq X$. For $x\in X$, define $d(x,A)=\inf\{d(x,a) : a\in A\}$. If $d(x,A)=0$ for all $x\in X$, then which of the following assertions must be true?

(1) A is compact (2) A is closed
(3) A is dense in X (4) $A=X$

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35. Let n be a positive integer and let H_n be the space of all $n \times n$ matrices $A = (a_{ij})$ with entries in $\mathbb{R}$ satisfying $a_{ij} = a_{rs}$ whenever $i + j = r + s$ $(i, j, r, s = 1, \ldots, n)$. Then the dimension of H_n, as a vector space over $\mathbb{R}$, is

(1) n^2 (2) $n^2 - n + 1$

(3) $2n + 1$ (4) $2n - 1$

36. Let $\underset{\sim}{X}$ be a p dimensional random vector which follows $N(\underset{\sim}{0}, I_p)$ distribution and let A be a real symmetric matrix. Which of the following is true?

1. $\underset{\sim}{X}^T A \underset{\sim}{X}$ has a chi-square distribution if $A^2 = A$ but the converse is not true
2. If $\underset{\sim}{X}^T A \underset{\sim}{X}$ has a chi-square distribution then the degrees of freedom is equal to p
3. If $\underset{\sim}{X}^T A \underset{\sim}{X}$ has a chi-square distribution then the characteristic roots of A are either 0 or 1
4. If $\underset{\sim}{X}^T A \underset{\sim}{X}$ has a chi-square distribution then A is necessarily positive definite

37. For which of the following primes p, does the polynomial $x^4 + x + 6$ have a root of multiplicity > 1 over a field of characteristic p?

(1) $p = 2$ (2) $p = 3$

(3) $p = 5$ (4) $p = 7$

38. Suppose $X \sim N(0,1)$ and $Y \sim \chi^2_n$. Which of the following is always correct?

1. $X^2 + Y \sim \chi^2_{n+1}$
2. $\dfrac{X}{\sqrt{Y/n}} \sim t_n$
3. $E(X^2 + Y) = 1 + n$
4. $Var(X + Y) = 1 + 2n$

39. The number of multiples of 10^{44} that divide 10^{55} is

(1) 11 (2) 12

(3) 121 (4) 144

40. Let $D = \{z \in \mathbb{C} : |z| < 1\}$ and let $f_n : D \to \mathbb{C}$ be defined by $f_n(z) = \dfrac{z^n}{n}$ for $n = 1, 2, \ldots$. Then

1. the sequences $\{f_n(z)\}$ and $\{f_n'(z)\}$ converge uniformly on D
2. the series $\sum_{n=1}^{\infty} f_n(z)$ converges uniformly on D
3. the series $\sum_{n=1}^{\infty} f_n'(z)$ converges for each $z \in D$
4. the sequence $\{f_n''(z)\}$ does not converge unless $z = 0$

41. Let $y_1(x)$ and $y_2(x)$ form a fundamental set of solutions of

$$\frac{d^2y}{dx^2} + p(x)\frac{dy}{dx} + q(x)y = 0, \quad a \le x \le b,$$

where $p(x)$ and $q(x)$ are real valued continuous functions on $[a, b]$. If x_0 and x_1, with $x_0 < x_1$, are consecutive zeros of $y_1(x)$ in (a, b), then

1. $y_1(x) = (x - x_0) q_0(x)$, where $q_0(x)$ is continuous on [a, b] with $q_0(x_0) \ne 0$
2. $y_1(x) = (x - x_0)^2 p_0(x)$, where $p_0(x)$ is continuous on [a, b] with $p_0(x_0) \ne 0$
3. $y_2(x)$ has no zeros in (x_0, x_1)
4. $y_2(x_0) = 0$ but $y_2'(x_0) \ne 0$

42. The number of group homomorphisms from the symmetric group S_3 to $\mathbb{Z}/6\mathbb{Z}$ is

(1) 1 **(2)** 2
(3) 3 **(4)** 6

43. The second order partial differential equation

$$\frac{(x-y)^2}{4}\frac{\partial^2 u}{\partial x^2}+(x-y)\sin\left(x^2+y^2\right)\frac{\partial^2 u}{\partial x\partial y}$$

$$+\cos^2\left(x^2+y^2\right)\frac{\partial^2 u}{\partial y^2}+(x-y)\frac{\partial u}{\partial x}$$

$$+\sin^2\left(x^2+y^2\right)\frac{\partial u}{\partial y}+u=0 \text{ is}$$

1. Elliptic in the region $\left\{(x,y):x\neq y,\ x^2+y^2<\pi/6\right\}$
2. Hyperbolic in the region $\left\{(x,y):x\neq y,\ \frac{\pi}{4}<x^2+y^2<\frac{3\pi}{4}\right\}$
3 Elliptic in the region $\left\{(x,y):x\neq y,\ \frac{\pi}{4}<x^2+y^2<\frac{3\pi}{4}\right\}$
4. Hyperbolic in the region $\left\{(x,y):x\neq y,\ x^2+y^2<\frac{\pi}{4}\right\}$

44. Let $X_1, X_2, \ldots$ be i.i.d. nonnegative integer valued random variables with finite mean and let N be an independent positive integer valued random variable with finite mean. Let $G(s)=E\left(s^{X_1}\right)$ and $H(s)=E\left(s^N\right)$ for $|s|<1$ and $G'(s), H'(s)$ denote the derivatives of $G(s)$, $H(s)$ respectively.

For $Y=X_1+X_2+\ldots+X_N$, the mean of Y is given by

1. $H'(G(0))G'(0)$
2. $G'(H(0))H'(0)$
3. $\lim_{s\uparrow 1} H'(G(s))G'(s)$
4. $\lim_{s\uparrow 1} G'(H(1-s))H'(s)$

45. Let $u=u(x,y)$ be the complete integral of the PDE $\frac{\partial u}{\partial x}\cdot\frac{\partial u}{\partial y}=xy$ passing through the points $(0,0,1)$ and $\left(0,1,\frac{1}{2}\right)$ in the $x-y-u$ space.

Then the value of $u(x,y)$ evaluated at $(-1,1)$ is

(1) 0 **(2)** 1
(3) 2 **(4)** 3

46. An aperiodic Markov chain with stationary transition probabilities on the state space $\{1,2,3,4,5\}$ must have

1. at least one null recurrent state
2. at least one positive recurrent state
3. at least one positive recurrent and at least one null recurrent state
4. at least one transient state

47. Consider a randomized (complete) block experiment involving v treatments and r replicates. Which of the following is true?

1. The variance of the best linear unbiased estimator (BLUE) of any elementary treatment contrast is σ^2/r, where σ^2 is the variance of an observation
2. The BLUE of a treatment contrast is uncorrelated with the BLUE of a block contrast
3. There are exactly $(r-1)$ linearly independent linear functions of the observations, each of which has zero expectation
4. If μ is the general mean and τ_i is the effect of the ith treatment $(i=1,2,\cdots,v)$, then $\mu+\tau_i$ is estimable for each $i, i=1,2,\cdots,v$

48. The complete integral of the PDE

$$\frac{\partial^2 u}{\partial x^2}+2\frac{\partial^2 u}{\partial x\partial y}+\frac{\partial^2 u}{\partial y^2}=xe^{x+y}$$

involving arbitrary functions ϕ_1 and ϕ_2 is

1. $\phi_1(y+x)+\phi_2(y+x)+\frac{1}{4}e^{x+y}$
2. $\phi_1(y+x)+x\phi_2(y+x)+\frac{(x-1)}{4}e^{x+y}$
3. $\phi_1(y-x)+\phi_2(y-x)+\frac{1}{4}e^{x+y}$
4. $\phi_1(y-x)+x\phi_2(y-x)+\frac{(x-1)}{4}e^{x+y}$

49. Suppose any two points $P(x_0,y_0)$, $Q(x_1,y_1)$, and a function $F(x,y,y')$ of three independent variables are given, where $y'\equiv\frac{dy}{dx}$. In order to find among all curves $y=y(x)$ joining P and Q that one which furnishes for the definite integral $I(y)=\int_{x_0}^{x_1}F(x,y(x),y'(x))\,dx$ the smallest value, which of the following assumption suffices:

1. Function F is of class C^1
2. Functional F is of class C^2 for all systems of values $x, y(x)$ and $y'(x)$ furnished by all of the admissible functions
3. Functional F is of class C^3 for all systems of values $x, y(x)$ and $y'(x)$ furnished by all of the admissible functions
4. It is enough to treat $y(x)$ and F to be of class C^1 only with respect to their arguments.

50. Let X_1, X_2 be a random sample from an exponential distribution with mean θ. Consider the problem of testing $H_0:\theta=1$ against $H_1:\theta=\frac{1}{2}$. Suppose the p-values of the left-tailed tests based on (X_1+X_2) and X_1 are given by p_1 and p_2 respectively. Which of the following is always true?

1. $p_1<p_2$
2. $p_1>p_2$
3. $p_1=p_2$
4. Nothing can be said about the relationship between p_1 and p_2

51. Consider the following linear programming problem
$Max\, Z=2x_1+x_2+x_3$
subject to

$$x_1-x_2\le 10$$
$$2x_1-x_2\le 40$$
$$x_1-x_3\le 10$$
$$x_1,\ x_2,\ x_3\ge 0.$$

The problem has

1. an unbounded solution
2. one optimal solution
3. more than one optional solutions
4. no feasible solution

52. Consider the equation
$\frac{dy}{dt}=(1+f^2(t))y(t),\ y(0)=1:t\ge 0$
where f is a bounded continuous function on $[0,\infty)$. Then

1. This equation admits a unique solution $y(t)$ and further $\lim_{t\to\infty}y(t)$ exists and is finite
2. This equation admits two linearly independent solutions
3. This equation admits a bounded solution for which $\lim_{t\to\infty}y(t)$ does not exist
4. This equation admits a unique solution $y(t)$ and further, $\lim_{t\to\infty}y(t)=\infty$

53. Let a sample of size n be drawn from a finite population of size N using probability proportional to size sampling ***with replacement***, the selection probabilities being

p_i, $1 \le i \le N, 0 < p_i < 1$ and $\sum_{i=1}^{N} p_i = 1$.

Also, let y_i be the value of a study variable for the ith unit in the sample $(1 \le i \le n)$ and $T = \sum_{i=1}^{n} y_i / (np_i)$. Which of the following is true?

1. The probability that the ith unit is included in the sample is p_i / n, $1 \le i \le N$
2. The variance of T is zero if $p_i = 1/N$ for each i, $1 \le i \le N$
3. The variance of T is zero if Y_i is proportional to p_i, where for $1 \le i \le N$, Y_i is the value of the study variable for the ith unit in the population
4. There is no unbiased estimator of the variance of T

54. For the computation of $\sqrt{x+1}-1$ at $x = 1.2345678 \times 10^{-5}$ using a machine which keeps 8 significant digits, which of the following equivalent expressions would be best to use

1. $\sqrt{x+1}-1$
2. $\left(1-\frac{1}{\sqrt{x-1}}\right)\sqrt{x+1}$
3. $\frac{x}{\sqrt{x+1}+1}$
4. $\frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 ....$

55. Let U denote the set of all $n \times n$ matrices A with complex entries such that A is unitary (i.e., $\overline{A}^t A = I_n$). Then U, as a topological subspace of $\mathbb{C}^{n^2}$, is

1. compact, but not connected
2. connected, but not compact
3. connected and compact
4. neither connected nor compact

56. Which of the following is correct?

1. The number of degrees of freedom of a two-particle (connected by a light rod of length L) system in a vertical plane, where one of the particles is constrained to move horizontally, is two
2. The number of degrees of freedom of a door swinging on its hinges is three
3. The number of independent generalized coordinates needed to specify the simple pendulum moving in a vertical plane is two
4. The number of scalar equations needed to determine the motion of an unconstrained N-particle system is N

57. A system has three components and the system works if at least two of the three components work. The life times of the components are independent and identically distributed exponential random variables with mean 1. If X denotes the life time of the system, then E(X) is

(1) 1 (2) 2/3
(3) 5/6 (4) 1/2

58. Let $X_1, \ldots, X_n$ be a random sample from the probability density function

$$f(x) = \lambda e^{-\lambda x}; x > 0, \lambda > 0,$$

where λ has the prior density

$$g(\lambda) = 9\lambda e^{-3\lambda}; \lambda > 0.$$

The Bayes' estimate of $P[1 < X_1 < 2]$ based on squared error loss function is

1. $\left[\frac{3+\Sigma x_i}{4+\Sigma x_i}\right]^n - \left[\frac{3+\Sigma x_i}{5+\Sigma x_i}\right]^n$
2. $\left[\frac{3+\Sigma x_i}{4+\Sigma x_i}\right]^{n+1} - \left[\frac{3+\Sigma x_i}{5+\Sigma x_i}\right]^{n+1}$

3. $\left[\frac{3+\Sigma x_i}{4+\Sigma x_i}\right]^{n+2} - \left[\frac{3+\Sigma x_i}{5+\Sigma x_i}\right]^{n+2}$

4. $\left[\frac{3+\Sigma x_i}{4+\Sigma x_i}\right]^{n+3} - \left[\frac{3+\Sigma x_i}{5+\Sigma x_i}\right]^{n+3}$

59. In a testing of hypothesis problem, the density of a sufficient statistic T is $f(t,\theta)=\frac{\theta}{t^{\theta+1}}, t>1, \theta>0$. The hypothesis $H_0 : \theta=1$ against $H_1 : \theta=2$ is to be tested and $T=2.5$ is observed. Then the p-value of the most powerful test is

(1) 0.05 **(2)** 0.5
(3) 0.6 **(4)** 0.4

60. Let Y_1, Y_2, Y_3 and Y_4 be four random variables such that
$E(Y_1)=\theta_1-\theta_3; E(Y_2)=\theta_1+\theta_2-\theta_3;$
$E(Y_3)=\theta_1-\theta_3; E(Y_4)=\theta_1-\theta_2-\theta_3,$
where $\theta_1, \theta_2, \theta_3$ are unknown parameters. Also assume that $Var(Y_i)=\sigma^2$, $i=1,2,3,4$. Then

1. $\theta_1, \theta_2, \theta_3$ are estimable
2. $\theta_1+\theta_3$ is estimable
3. $\theta_1-\theta_3$ is estimable and $\frac{1}{2}(Y_1+Y_3)$ is the best linear unbiased estimate of $\theta_1-\theta_3$
4. θ_2 is estimable

PART C

__Unit - I__

61. Which of the following real-valued functions f defined on $\mathbb{R}$ have the property that f is continuous and $f \circ f = f$?

(1) $f(x)=\begin{cases}|x| & \text{if } x\in[-1,1],\\ x^2 & \text{if } x\notin[-1,1]\end{cases}$

(2) $f(x)=\begin{cases}x & \text{if } x\in[0,1],\\ 1 & \text{if } x\geq 1,\\ 0 & \text{if } x\leq 0\end{cases}$

(3) $f(x)=\begin{cases}x & \text{if } x\in[-1,1],\\ 1 & \text{if } x\geq 1,\\ -1 & \text{if } x\leq -1\end{cases}$

(4) $f(x)=\begin{cases}1 & \text{if } x\in[-23,27],\\ 22+x & \text{if } x\leq -23,\\ -26+x & \text{if } x\geq 27\end{cases}$

62. Let $f:\mathbb{R}\to\mathbb{R}$ be a continuous function such that $\int_0^\infty f(x)\,dx$ exists. Which of the following statements are correct?

1. if $\lim_{x\to\infty} f(x)$ exists, then $\lim_{x\to\infty} f(x)=0$
2. the limit $\lim_{x\to\infty} f(x)$ must exist and is zero
3. in case f is a nonnegative function, $\lim_{x\to\infty} f(x)$ must exist and is zero
4. in case f is a differentiable function, $\lim_{x\to\infty} f'(x)$ must exist and is zero

63. Let $f(r,\theta)=(r\cos\theta, r\sin\theta)$ for $(r,\theta)\in\mathbb{R}^2$ with $r\neq 0$. Which of the following statements are correct? (Here Df denotes the derivative of f).

1. the linear transformation $Df(r,\theta)$ is not zero for any $(r,\theta)\in\mathbb{R}^2$ with $r\neq 0$
2. f is one-one on $\{(r,\theta)\in\mathbb{R}^2 : r\neq 0\}$
3. for any $(r,\theta)\in\mathbb{R}^2$ with $r\neq 0$, f is one-one on a neighborhood of (r,θ)
4. $Df(r,\theta)=r^2 I$ for any $(r,\theta)\in\mathbb{R}^2$ with $r\neq 0$

64. The map $L:\mathbb{R}^2\to\mathbb{R}^2$ given by $L(x,y)=(x,-y)$ is

(1) differentiable everywhere on $\mathbb{R}^2$ (2) differentiable only at (0,0)

(3) $DL(0,0)=L$ (4) $DL(x,y)=L$ for all $(x,y)\in\mathbb{R}^2$

65. It is given that the series $\sum_{n=1}^{\infty} a_n$ is convergent, but not absolutely convergent and $\sum_{n=1}^{\infty} a_n=0$. Denote by s_k the partial sum $\sum_{n=1}^{k} a_n$, $k=1,2,\cdots$ Then

1. $s_k=0$ for infinitely many k
2. $s_k>0$ for infinitely many k, and $s_k<0$ for infinitely many k
3. it is possible that $s_k>0$ for all k
4. it is possible that $s_k>0$ for all but a finite number of values of k

66. Define $f:\mathbb{R}\to\mathbb{R}$ by $f(x)=\begin{cases} x^2 & \text{if } x<0, \\ 2x+x^2 & \text{if } x\geq 0 \end{cases}$

Then which of the following statements are correct?

(1) $f''(x)=2$ for all $x\in\mathbb{R}$ (2) $f''(0)$ does not exist

(3) $f'(x)$ exists for each $x\neq 0$ (4) $f'(0)$ does not exist

67. Let $f:\mathbb{R}\to\mathbb{R}$ be a differentiable function such that f' is bounded. Given a closed and bounded interval $[a,b]$, and a partition $P=\{a_0=a<a_1<\cdots<a_n=b\}$ of $[a,b]$, let $M(f,P)$ and $m(f,P)$ denote, respectively, the upper Riemann sum and the lower Riemann sum of f with respect to P. Then

1. $\left|M(f,P)-\int_a^b f(x)\,dx\right| \le (b-a)\sup\{|f(x)| : x\in[a,b]\}$

2. $\left|m(f,P)-\int_a^b f(x)\,dx\right| \le (b-a)\inf\{|f(x)| : x\in[a,b]\}$

3. $\left|M(f,P)-\int_a^b f(x)\,dx\right| \le (b-a)^2\sup\{|f'(x)| : x\in[a,b]\}$

4. $\left|m(f,P)-\int_a^b f(x)\,dx\right| \le (b-a)^2\inf\{|f'(x)| : x\in[a,b]\}$

68. Let $f:\mathbb{R}^n\to\mathbb{R}^n$ be the function defined by $f(x)=x\|x\|^2$ for $x\in\mathbb{R}^n$. Which of the following statements are correct?

(1) $(Df)(0)=0$

(2) $(Df)(x)=0$ for all $x\in\mathbb{R}^n$

(3) f is one-one

(4) f has an inverse

69. Consider the quadratic form $q(x,y,z)=4x^2+y^2-z^2+4xy-2xz-yz$ over $\mathbb{R}$. Which of the following statements about the range of values taken by q as x,y,z vary over $\mathbb{R}$, are true?

1. range contains $[1,\infty)$
2. range is contained in $[0,\infty)$
3. range $=\mathbb{R}$
4. range is contained in $[-N,\infty)$ for some large natural number N depending on q

70. Consider a matrix $A=(a_{ij})_{n\times n}$ with integer entries such that

$$a_{ij}=0 \text{ for } i>j \text{ and } a_{ii}=1 \text{ for } i=1,\dots,n.$$

Which of the following properties must be true?

1. A^{-1} exists and it has integer entries
2. A^{-1} exists and it has some entries that are not integers
3. A^{-1} is a polynomial function of A with integer coefficients
4. A^{-1} is not a power of A unless A is the identity matrix

71. Let J be the 3×3 matrix all of whose entries are 1. Then:

1. 0 and 3 are the only eigenvalues of A
2. J is positive semidefinite, i.e., $\langle Jx,x\rangle \geq 0$ for all $x \in \mathbb{R}^3$
3. J is diagonalizable
4. J is positive definite, i.e., $\langle Jx,x\rangle > 0$ for all $x \in \mathbb{R}^3$ with $x \neq 0$

72. Let A, B be complex $n\times n$ matrices. Which of the following statements are true?

1. If A, B, and $A+B$ are invertible, then $A^{-1}+B^{-1}$ is invertible
2. If A, B, and $A+B$ are invertible, then $A^{-1}-B^{-1}$ is invertible
3. If AB is nilpotent, then BA is nilpotent
4. Characteristic polynomials of AB and BA are equal if A is invertible

73. Let ω be a complex number such that $\omega^3=1$, but $\omega\neq1$. If

$$A = \begin{bmatrix} 1 & \omega & \omega^2 \\ \omega & \omega^2 & 1 \\ \omega^2 & \omega & 1 \end{bmatrix},$$

then which of the following statements are true?

1. A is invertible
2. $\operatorname{rank}(A)=2$
3. 0 is an eigenvalue of A
4. there exist linearly independent vectors $v, w \in \mathbb{C}^3$ such that $Av=Aw=0$

74. Let A be a 4×4 matrix with real entries such that $-1, 1, 2, -2$ are its eigenvalues. If $B=A^4-5A^2+5I$, where I denotes the 4×4 identity matrix, then which of the following statements are correct?

(1) $\det(A+B)=0$ (2) $\det(B)=1$

(3) trace of $A-B$ is 0 (4) trace of $A+B$ is 4

75. Let $M_2(\mathbb{R})$ denote the set of 2×2 real matrices. Let $A \in M_2(\mathbb{R})$ be of trace 2 and determinant -3. Identifying $M_2(\mathbb{R})$ with $\mathbb{R}^4$, consider the linear transformation $T: M_2(\mathbb{R}) \to M_2(\mathbb{R})$ defined by $T(B)=AB$. Then which of the following statements are true?

(1) T is diagonalizable (2) 2 is an eigenvalue of T

(3) T is invertible (4) $T(B)=B$ for some $0\neq B$ in $M_2(\mathbb{R})$

76. Let A be a 2×2 non-zero matrix with entries in $\mathbb{C}$ such that $A^2=0$. Which of the following statements must be true?

1. PAP^{-1} is diagonal for some invertible 2×2 matrix P with entries in $\mathbb{R}$
2. A has two distinct eigenvalues in $\mathbb{C}$
3. A has only one eigenvalue in $\mathbb{C}$ with multiplicity 2
4. $Av=v$ for some $v\in\mathbb{C}^2, v\neq 0$

77. Consider the linear transformation $T:\mathbb{R}^7\to\mathbb{R}^7$ defined by

$$T(x_1,x_2,\dots,x_6,x_7)=(x_7,x_6,\dots,x_2,x_1).$$

Which of the following statements are true?

1. the determinant of T is 1
2. there is a basis of $\mathbb{R}^7$ with respect to which T is a diagonal matrix
3. $T^7=I$
4. The smallest n such that $T^n=I$ is even

78. Let λ,μ be distinct eigenvalues of a 2×2 matrix A. Then, which of the following statements must be true?

1. A^2 has distinct eigenvalues
2. $A^3 = \dfrac{\lambda^3-\mu^3}{\lambda-\mu}A - \lambda\mu(\lambda+\mu)I$
3. trace of A^n is $\lambda^n+\mu^n$ for every positive integer n
4. A^n is not a scalar multiple of identity for any positive integer n

Unit - II

79. Let f be an entire function such that $\lim\limits_{|z|\to\infty}|f(z)|=\infty$. Then

1. $f\left(\frac{1}{z}\right)$ has an essential singularity at 0.
2. f cannot be a polynomial
3. f has finitely many zeros
4. $f\left(\frac{1}{z}\right)$ has a pole at 0

80. Let f be a holomorphic function on $D=\{z\in\mathbb{C}:|z|<1\}$ such that $|f(z)|\le 1$ Define $g:D\to\mathbb{C}$ by

$$g(z)=\begin{cases}\frac{f(z)}{z} & \text{if } z\in D, z\neq 0,\\ f'(0) & \text{if } z=0.\end{cases}$$

Which of the following statements are true?

(1) g is holomorphic on D (2) $|g(z)|\le 1$ for all $z\in D$

(3) $|f'(z)|\le 1$ for all $z\in D$ (4) $|f'(0)|\le 1$

81. Let f,g be holomorphic functions defined on $A\cup D$, where

$$A = \left\{z\in\mathbb{C} : \frac{1}{2}<|z|<1\right\} \text{ and}$$
$$D = \{z\in\mathbb{C} : |z-2|<1\}.$$

Which of the following statements are correct?

1. if $f(z)g(z)=0$ for all $z\in A\cup D$, then either $f(z)=0$ for all $z\in A$ or $g(z)=0$ for all $z\in A$
2. if $f(z)g(z)=0$ for all $z\in D$, then either $f(z)=0$ for all $z\in D$ or $g(z)=0$ for all $z\in D$
3. if $f(z)g(z)=0$ for all $z\in A$, then either $f(z)=0$ for all $z\in A$ or $g(z)=0$ for all $z\in A$
4. if $f(z)g(z)=0$ for all $z\in A\cup D$, then either $f(z)=0$ for all $z\in A\cup D$, or $g(z)=0$ for all $z\in A\cup D$.

82. Let $\mathbb{Z}[i]$ denote the ring of Gaussian integers. For which of the following values of n is the quotient ring $\mathbb{Z}[i]/n\mathbb{Z}[i]$ an integral domain?

(1) 2 (2) 13 (3) 19 (4) 7

83. Let U be an open subset of $\mathbb{C}$ containing $\{z\in\mathbb{C}:|z|\le 1\}$ and let $f:U\to\mathbb{C}$. be the map defined by

$$f(z)=e^{i\psi}\frac{z-a}{1-\bar{a}z} \text{ for } a\in D, \text{and } \psi\in[0.2\pi].$$

Which of the following statements are true?

(1) $|f(e^{i\theta})|=1$ for $0\le\theta\le 2\pi$ (2) f maps $\{z\in\mathbb{C}:|z|<1\}$ onto itself

(3) f maps $\{z\in\mathbb{C}:|z|\le 1\}$ into itself (4) f is one-one

84. Which of the following integral domains are Euclidean domains?

1. $\mathbb{Z}\left[\sqrt{-3}\right]=\left\{a+b\sqrt{-3} : a,b\in\mathbb{Z}\right\}$
2. $\mathbb{Z}[x]$
3. $\mathbb{R}\left[x^2.x^3\right]=\left\{f=\sum_{i=0}^{n}a_ix^i\in\mathbb{R}[x] : a_1=0\right\}$
4. $\left(\frac{\mathbb{Z}[x]}{(2,x)}\right)[y]$ where x, y are independent variables and $(2, x)$ is the ideal generated by 2 and x

85. Let $f:\mathbb{C}\to\mathbb{C}$ be an entire function and let $g:\mathbb{C}\to\mathbb{C}$ be defined by $g(z)=f(z)-f(z+1)$ for $z\in\mathbb{C}$. Which of the following statements are true?

1. if $f\left(\frac{1}{n}\right)=0$ for all positive integers n, then f is a constant function
2. if $f(n)=0$ for all positive integers n, then f is a constant function
3. if $f\left(\frac{1}{n}\right)=f\left(\frac{1}{n}+1\right)$ for all positive integers n, then g is a constant function
4. if $f(n)=f(n+1)$ for all positive integers n, then g is a constant function

86. For $x\in\mathbb{R}^n$, let $B(x,r)$ denote the closed ball in $\mathbb{R}^n$ (with the Euclidean norm) of radius r centered at x. Write $B=B(0,1)$. If $f, g:\mathrm{B}\to\mathbb{R}^n$ are continuous functions such that $f(x)\neq g(x)$ for all $x\in B$, then

1. $f(B)\cap g(B)=\varnothing$
2. there exists $\varepsilon>0$ such that $\|f(x)-g(x)\|>\varepsilon$ for all $x\in B$
3. there exists $\varepsilon>0$ such that $B(f(x),\varepsilon)\cap B(g(x),\varepsilon)=\varnothing$ for all $x\in B$
4. $\mathrm{Int}(f(B))\cap\mathrm{Int}(g(B))=\varnothing$, where Int(E) denotes the interior of a set E.

87. Let G be the Galois group of the splitting field of x^5-2 over $\mathbb{Q}$. Then, which of the following statements are true?

(1) G is cyclic
(2) G is non-abelian
(3) the order of G is 20
(4) G has an element of order 4

88. For which of the following values of n, does the finite field F_{5^n} with 5^n elements contain a non-trivial 93^{rd} root of unity?

(1) 92 (2) 30 (3) 15 (4) 6

89. Let X denote the product of countably many copies of $[0,1]$. We let X_1 denote the set X equipped with the box topology and let X_2 denote the set X equipped with the product topology. Then

(1) X_1 is compact and separable (2) X_2 is compact and separable

(3) X_1 and X_2 are both compact (4) Neither X_1 nor X_2 is separable

90. Which of the following numbers can be orders of permutations σ of 11 symbols such that σ does not fix any symbol?

(1) 18 (2) 30 (3) 15 (4) 28

Unit - III

91. Consider the functional

$$I(y)=\int_{1/2}^{x_1}\sqrt{1+y'^2}\,dx$$

subject to the condition that the left end of the extremal is fixed at the point $\left(\frac{1}{2},\frac{1}{4}\right)$ and the right end (x_1,y_1) be movable along the straight line $y=x-5$. Let C_1 and C_2 be arbitrary constants. Then

1. the general solution of the Euler's equation satisfies $2C_1+4C_2=1$
2. the transversality condition that holds at the right end yields
$$\sqrt{\left(1+C_1^2\right)}+\left(1-C_1\right)\frac{C_1}{\sqrt{1+C_1^2}}=0$$
3. the relation $C_1x_1+C_2=x_1-5$ is true
4. the extremal is $y=-x+\frac{3}{4}$

92. For the integral equation $u(x)=f(x)+\lambda\int_a^b K(x,t)u(t)\,dt$ to have a continuous solution in the interval $a\le x\le b$, which of the following assumptions are necessary?

1. $K(x,t)\neq 0$, is real and continuous in the region $a\le x\le b$, $a\le t\le b$ with $|K(x,t)|\le M$
2. $f(x)\neq 0$, is real and continuous in the interval $a\le x\le b$,

3. λ is a constant
4. $|\lambda| < \frac{1}{M(b-a)}$

93. Suppose that the potential energy $V(q_1, q_2)$ of a system with generalized coordinates q_1 and q_2 has a minimum at $q_1 = q_2 = 0$ and that $V(0,0) = 0$. Then

1. the approximate V near (0, 0) is a homogeneous quadratic form in q_1 and q_2 assuming that the terms in powers three (or higher) in the small quantities q_1 and q_2 are negligible
2. if (0,0) is also a minimum point of approximate V, then the quadratic form in the option 1 given above is positive definite
3. the requirement in the option 2 given above implies that the approximate V takes positive values except when $q_1 = q_2 = 0$
4. the requirement in the option 2 given above that (0,0) is a minimum point of approximate V implies that $q_1 = q_2 = 0$ yields a strict minimum of the exact V

94. The integral equation

$y(x) = 1 + \lambda \int_0^{\pi/2} \cos(x-t)\, y(t)\, dt$ has

1. a unique solution for $\lambda \neq 4/(\pi+2)$
2. a unique solution for $\lambda \neq 4/(\pi-2)$
3. no solution for $\lambda = 4/(\pi+2)$, but the corresponding homogeneous equation has non-trivial solutions
4. no solution for $\lambda = 4/(\pi-2)$, but the corresponding homogenous equation has non-trivial solutions

95. Consider the Euler's equations of motion for the free motion of a body relative to its centre of mass O and principal axes OX_1, OX_2, OX_3. If the body is in motion with OX_3 as the axis of symmetry with $\bar{\omega} = (\omega_1, \omega_2, \omega_3)$ and $\bar{L} = (L_1, L_2, L_3)$ as the angular velocity vector and the angular momentum vector, respectively, then

1. ω_3 = constant
2. ω_1 and ω_2 satisfy the simple harmonic motion equation
3. $\bar{e}_3$, the unit vector along OX_3, $\bar{\omega}$, and $\bar{L}$ lie in the same plane
4. L_3 = constant

96. Consider the first order system of linear equations

$\frac{dX}{dt} = AX;\ A = \begin{bmatrix} 3 & 2 \\ -2 & -1 \end{bmatrix};\ X = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}.$

Then

1. the coefficient matrix A has a repeated eigenvalue $\lambda = 1$
2. there is only one linearly independent eigenvector $X_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$
3. the general solution of the ODE is $(aX_1 + bX_2)e^t$, where a, b are arbitrary constants and $X_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$, $X_2 = \begin{bmatrix} t \\ \frac{1}{2} - t \end{bmatrix}$
4. the vectors X_1 and X_2 in the option 3 given above are linearly independent

97. Consider the interpolation data given below:

x	1	½	3
y	3	−10	2

The interpolating polynomial corresponding to this data is given by

1. $p(x) = -3\left(x - \frac{1}{2}\right)(x-3) - 8(x-1)(x-3) + \frac{2}{5}(x-1)\left(x - \frac{1}{2}\right)$
2. $q(x) = 3 + 26(x-1) + \frac{-53}{5}(x-1)\left(x - \frac{1}{2}\right)$
3. $r(x) = \frac{-53}{5}x^2 + \frac{419}{10}x + \frac{-283}{10}$
4. $p(x)q(x) + r(x)$

98. The Green's function G(x,t) of the boundary value problem

$$\frac{d^2y}{dx^2} - \frac{1}{x}\frac{dy}{dx} = 1, \quad y(0) = y(1) = 0$$

is $G(x,t) = \begin{cases} f_1(x,t), & \text{if } x \le t \\ f_2(x,t), & \text{if } t \le x, \end{cases}$

where

1. $f_1(x,t) = -\frac{1}{2}t(1-x^2)$, $f_2(x,t) = -\frac{1}{2t}x^2(1-t^2)$
2. $f_1(x,t) = -\frac{1}{2x}t^2(1-x^2)$, $f_2(x,t) = -\frac{1}{2t}x^2(1-t^2)$
3. $f_1(x,t) = -\frac{1}{2t}x^2(1-t^2)$, $f_2(x,t) = -\frac{1}{2}t(1-x^2)$
4. $f_1(x,t) = -\frac{1}{2t}x^2(1-t^2)$, $f_2(x,t) = -\frac{1}{2x}t^2(1-x^2)$

99. Let $A=\begin{pmatrix}1.01 & 0.99\\ 0.99 & 1.01\end{pmatrix}$, $b=\begin{pmatrix}2\\2\end{pmatrix}$ and x_0 be the unique solution of the equation $Ax=b$,

Let $\hat{x}_1$ and $\hat{x}_2$ be the two approximate solutions $\begin{pmatrix}1.01\\1.01\end{pmatrix}$ and $\begin{pmatrix}2\\0\end{pmatrix}$.

Finally, let $r_1 = Ax_0 - A\hat{x}_1 = b - A\hat{x}_1$ and $r_2 = Ax_0 - A\hat{x}_2 = b - A\hat{x}_2$ be the corresponding residues. Which of the following is/are correct?

1. $\hat{x}_1$ is a good approximation to x_0 and r_1 is small
2. $\hat{x}_1$ is not a good approximation to x_0 but r_1 is small
3. $\hat{x}_2$ is a good approximation to x_0 and r_2 is small
4. $\hat{x}_2$ is not a good approximation to x_0 but r_2 is small

100. Let $u(x,t)$ be the solution of the initial boundary value problem

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}, \quad 0<x<\infty, t>0$$

$$u(x,0)=\cos\left(\frac{\pi x}{2}\right), \quad 0\le x<\infty$$

$$\frac{\partial u}{\partial t}(x,0)=0, \quad 0\le x<\infty,$$

$$\frac{\partial u}{\partial x}(0,t)=0, \quad t\ge 0.$$

Then

(1) The value of $u(2,2)=-1$ (2) The value of $u(2,2)=1$

(3) The value of $u\left(\frac{1}{2},\frac{1}{2}\right)=\frac{1}{\sqrt{2}}$ (4) The value of $u\left(\frac{1}{2},\frac{1}{2}\right)=\frac{1}{2}$

101. The differential equation

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=2u$$

satisfying the initial condition $y=xg(x)$, $u=f(x)$ with

1. $f(x)=2x$, $g(x)=1$, has no solution
2. $f(x)=2x^2$, $g(x)=1$, has infinite number of solutions
3. $f(x)=x^3$, $g(x)=x$, has a unique solution
4. $f(x)=x^4$, $g(x)=x$, has a unique solution

102. Let $\frac{d^2y}{dx^2} - q(x)y = 0, \quad 0 \le x < \infty,$

$y(0) = 1, \frac{dy}{dx}(0) = 1,$ where $q(x)$ is a positive monotonically increasing continuous function. Then

1. $y(x) \to \infty$ as $x \to \infty$
2. $\frac{dy}{dx} \to \infty$ as $x \to \infty$
3. $y(x)$ has finitely many zeros in $[0, \infty)$
4. $y(x)$ has infinitely many zeros in $[0, \infty)$

Unit - IV

103. Consider a multiple linear regression model with r regressors, $r \ge 1$ and the response variable Y. Suppress $\hat{Y}$ is the fitted value of Y, R^2 is the coefficient of determination and R^2_{adj} is the adjusted coefficient of determination. Then

1. R^2 always increases if an additional regressor is included in the model
2. R^2_{adj} always increases if an additional regressor is included in the model
3. $R^2 \ge R^2_{\text{adj}}$ for all r
4. correlation coefficient between Y and $\hat{Y}$ is always non-negative

104. Let $X_1, X_2 \cdots X_n$ be independent and identically distributed random variables with common mean μ and finite variance σ^2. Define $S_n = \sqrt{\frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X}_n)^2}$ where $\bar{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i$. Then which of the following is/are true?

(1) S_n is unbiased for estimating σ (2) S_n^2 is unbiased for estimating σ^2
(3) S_n is consistent for estimating σ (4) S_n^2 is consistent for estimating σ^2

105. The statement

"X_n converges to X in distribution"

is equivalent to

1. $\limsup_{n\to\infty} P[X_n \le x] \le P[X \le x]$ for all real x
2. $\liminf_{n\to\infty} P[X_n < x] \ge P[X < x]$ and $\liminf_{n\to\infty} P[X_n > x] \ge P[X > x]$ for all real x

3. $E\left[g(X_n)\right] \to \left[g(X)\right]$ for all bounded continuous functions g

4. $E\left[g(X_n)\right] \to E\left[g(X)\right]$ for all uniformly continuous functions g

106. An experiment consists of independent trials. At the n th trial a number is chosen uniformly at random from $\{1, 2, \ldots, n\}$. A random variable T_n is defined as follows

$$T_n = \begin{cases} 1, & \text{if } n \text{ is chosen at the } n\text{th trial} \\ 0, & \text{otherwise.} \end{cases}$$

Then

1. $P\left(T_n = 1 \text{ for infinitely many } n \geq 1\right) = 1$
2. $P\left(T_n = 0 \text{ for infinitely many } n \geq 1\right) = 0$
3. $P\left(T_n = 1 \text{ and } T_{n+1} = 0 \text{ for infinitely many } n \geq 1\right) = 1$
4. $P\left(T_n = 1 \text{ and } T_{n+1} = 1 \text{ for infinitely many } n \geq 1\right) = 0$

107. Let $\left(X_{n1}, X_{n2}, X_{n3}, X_{n4}\right)$ have a multinomial distribution with probability mass function

$$P[X_{n1} = k_1, X_{n2} = k_2, X_{n3} = k_3, X_{n4} = k_4] = \frac{n!}{k_1!\,k_2!\,k_3!\,k_4!}\left(\tfrac{1}{n}\right)^{k_1}\left(\tfrac{1}{n}\right)^{k_2}\left(\tfrac{n-2}{2n}\right)^{k_3}\left(\tfrac{n-2}{2n}\right)^{k_4},$$

where k_1, k_2, k_3, k_4 are non-negative integers and $k_1 + k_2 + k_3 + k_4 = n$. Then

1. X_{n1} converges in distribution to a Poisson random variable with mean 1
2. $\left(X_{n1}, X_{n2}\right)$ converges in distribution to $\left(X_1, X_2\right)$ where X_1 and X_2 are <u>not</u> independent
3. $\left(X_{n1}, X_{n3}\right)$ converges in distribution to $\left(X_1, X_3\right)$ where X_1 and X_3 are independent
4. $\left(X_{n1}, X_{n2}, X_{n3}\right)$ does not converge in distribution

108. Let $X_1, X_2, \ldots$ be i.i.d random variables with mean 0 and variance 2 and let $Y_i = \dfrac{X_i + X_{i+1}}{2}$ for $i \geq 1$. Then the limiting distribution of

1. $\dfrac{1}{\sqrt{2n}}\left(Y_1 + Y_2 + \cdots + Y_n\right)$ is normal with mean 0 and variance 1
2. $\dfrac{1}{\sqrt{n}}\left(Y_1 + Y_3 + Y_5 + \cdots + Y_{2n-1}\right)$ is normal with mean 0 and variance 1
3. $\dfrac{1}{\sqrt{n}}\left(Y_2 + Y_4 + Y_6 + \cdots + Y_{2n}\right)$ is normal with mean 0 and variance 1
4. $\dfrac{1}{\sqrt{n}}\left(Y_1 + Y_{2^1} + Y_{2^2} + \cdots + Y_{2^{n-1}}\right)$ is normal with mean 0 and variance 1

109. Consider the following primal problem

Max $Z = 5x_1 + 12x_2 + 4x_3$

subject to

$x_1 + 2x_2 + x_3 \leq 10$

$2x_1 - x_2 + 3x_3 \leq 8$

x_1 unrestricted, $x_2, x_3 \geq 0$.

Then

1. the objective function of the dual problem is Min $W = 10y_1 + 8y_2$
2. two of the constraints of the dual problem are $y_1 + 2y_2 = 5$, $2y_1 - y_2 \geq 12$
3. one of the constraint of the dual problem is $2y_1 + 3y_2 \geq 4$
4. for a pair of feasible primal and dual solutions we have $v_1 \geq v_2$ where v_1 is the value of the objective function in the maximization problem and v_2 is the value of the objective function in the minimization problem

110. For a random variable X with probability density function

$$f(x) = \left(1 + \alpha x^{\alpha-1}\right) e^{-\left(x + x^{\alpha}\right)}, x > 0, \alpha > 0,$$

the hazard function can be

(1) constant for some α (2) an increasing function for some α

(3) a decreasing function for some α (4) a bathtub-shaped function for some α

111. Let d be a balanced incomplete block design with usual parameters v, b, r, k, λ and let $\tau_i (1 \leq i \leq v)$ be the effect of the i th treatment. Which of the following is/are true?

1. d is connected whenever $k \geq 2$
2. Let p_i $(1 \leq i \leq v)$ be real numbers satisfying $\sum_{i=1}^{v} p_i = 0$ and $\sum_{i=1}^{v} p_i^2 = 1$. The variance of the best linear unbiased estimator of $\sum_{i=1}^{v} p_i \tau_i$, under d, does not depend on the p_i's
3. For $1 \leq i \leq v$, let p_i and q_i be real numbers satisfying $\sum_{i=1}^{v} p_i = 0 = \sum_{i=1}^{v} q_i$ and $\sum_{i=1}^{v} p_i q_i = 0$. The covariance between the best linear unbiased estimators of $\sum_{i=1}^{v} p_i \tau_i$ and $\sum_{i=1}^{v} q_i \tau_i$, under d, is zero
4. For d, the inequality $b \geq v + r - k$ holds

112. Consider a finite population of $N=nk$ units, where $n(\geq 2)$, $k(\geq 2)$ are integers. A linear systematic sample of n units is drawn from the population. Which of the following is/are true?

1. The probability that the ith unit is included in the sample is $1/k$, $i=1,2,\cdots,N$
2. If π_{ij} denotes the probability of inclusion of the units i and j in the sample, $i\neq j$, $i, j = 1, 2, ..., N$, then π_{ij} is zero for some pairs (i, j)
3. An unbiased estimator of the population mean of a study variable is the sample mean
4. There always exists an unbiased estimator of the variance of the sample mean

113. Let X_1, X_2, X_3, X_4 and Y_1, Y_2, Y_3 be two independent random samples from the continuous distributions $F(x)$ and $F(x-\Delta)$ respectively.

Define Rank $(X_i)=R_i$, $i=1,2,3,4$ among $X_1, X_2, X_3, X_4, Y_1, Y_2, Y_3$. If the observed value of $\sum_{i=1}^{4} R_i$ is 11 and the Wilcoxon rank-sum test is used for testing $H_0:\Delta=0$ against $H_1:\Delta>0$, then which of the following is/are true?

(1) P-value < 0.06
(2) Reject H_0 at 5% level of significance
(3) Accept H_0 at 5% level of significance
(4) Accept H_0 at 1% level of significance

114. $(X_1,Y_1),(X_2,Y_2),\cdots,(X_n,Y_n)$ are independent and identically distributed random vectors where X_1 is normally distributed with mean 0 and variance 1 and $P[Y_1=2]=P[Y_1=-2]=\frac{1}{2}$. Further, X_i and Y_i are independently distributed for $i=1,..,n$ and $Z_n = X_1Y_1 - X_2Y_2 + \cdots + (-1)^{n-1} X_nY_n$. Then

(1) Z_n is normally distributed for each n
(2) Z_n is symmetrically distributed abo

(3) $V\left(\frac{Z_n}{\sqrt{n}}\right)=4$
(4) $V\left(\frac{Z_n}{\sqrt{n}}\right)=2$

115. Let Q_n denote the length of a queue at time n in an $M/M/1$ queue with arrival rate $\lambda>0$, service rate $\mu>0$ and $\rho=\frac{\lambda}{\mu}$. Which of the following is/are true?

1. If $\lambda<\mu$, then $\lim_{n\to\infty} P(Q_n=k)=(1-\rho)\rho^k$, $k\geq 0$
2. If $\lambda=\mu$, then $\lim_{n\to\infty} P(Q_n=k)=\frac{1}{2^{k+1}}$, $k\geq 0$
3. If $\lambda>\mu$, then $\lim_{n\to\infty} P(Q_n=k)=(1-\rho)\rho^k$, $k\geq 0$
4. If $\lambda=\mu$, then $\lim_{n\to\infty} P(Q_n=k)=0$, $k\geq 0$

116. Let P be the stationary transition probability matrix of the Markov Chain $\{X_n, n \geq 0\}$, which is irreducible and every state has period 2. Further suppose that the Markov chain $\{Y_n, n \geq 0\}$ on the same state space has transition probability matrix P^2. Both the chains are assumed to have the same initial distribution. Then

1. $P[X_0 = Y_0, X_2 = Y_1] = 1$
2. all states of the chain $\{Y_n, n \geq 0\}$ are aperiodic
3. the chain $\{Y_n, n \geq 0\}$ is irreducible
4. if a state is recurrent for the chain $\{X_n, n \geq 0\}$, then it is also recurrent for the chain $\{Y_n, n \geq 0\}$

117. $X_1, X_2, \cdots, X_n$ are independent and identically distributed random variables with a common p.d.f. parametrized by $\theta > 0$ given by

$$f(x) = \begin{cases} \dfrac{\theta}{x^2}, & x \geq \theta \\ 0, & \text{otherwise.} \end{cases}$$

Let $X_{min} = \min\{X_1, \ldots, X_n\}$ and $X_{max} = \max\{X_1, \ldots, X_n\}$. Then

1. X_{min} is sufficient for θ
2. $\dfrac{1}{\prod_{i=1}^{n} X_i^2}$ is sufficient for θ
3. X_{max} is maximum likelihood estimator of θ
4. X_{min} is maximum likelihood estimator of θ

118. Let $\underset{\sim}{\mu} = (\mu_1, \mu_2, \mu_3, \mu_4)$ be the stationary distribution for a Markov chain on the state space $\{1, 2, 3, 4\}$ with transition probability matrix P. Suppose that the states 1 and 2 are transient and the states 3 and 4 form a communicating class. Which of the following is/are true?

(1) $\underset{\sim}{\mu} P^3 = \underset{\sim}{\mu} P^5$
(2) $\mu_1 = 0$ and $\mu_2 = 0$
(3) $\mu_3 + \mu_4 = 1$
(4) One of μ_3 and μ_4 is zero.

119. Suppose $\underset{\sim}{X}$ is a p-dimensional random vector with variance-covariance matrix Σ. If $\underset{\sim}{P}_1, \cdots \underset{\sim}{P}_p$ represent p orthonormal eigenvectors of Σ corresponding to the eigenvalues $\lambda_1 > \lambda_2 > \cdots > \lambda_p \geq 0$ respectively, then which of the following is/are true?

(1) First principal component is $\underset{\sim}{P}_1^T \underset{\sim}{X}$
(2) $Var\left(\underset{\sim}{P}_1^T \underset{\sim}{X}\right) = \lambda_1$
(3) $\underset{\sim}{P}_1^T \underset{\sim}{X}$ and $\underset{\sim}{P}_2^T \underset{\sim}{X}$ are correlated
(4) $Tr(\Sigma) = \sum_{i=1}^{p} \lambda_i$

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120. Consider the problem of testing H_0:X~p_0 against H_1:X~p_1 where p_0 and p_1 are given by

x	0	1	2	3	4	5
$p_0(x)=P_{H_0}(X=x)$	0.05	0.02	0.03	0.05	0.35	0.5
$\frac{p_1(x)}{p_0(x)}=\frac{P_{H_1}(X=x)}{P_{H_0}(X=x)}$	2	1.5	1.5	3	0.4	1.07

Define three tests ϕ_1, ϕ_2 and ϕ_3 such that

$$\phi_1(x)=\begin{cases}1, & \text{if } x=0\\ 0, & \text{otherwise}\end{cases}$$

$$\phi_2(x)=\begin{cases}1, & \text{if } x=3\\ 0, & \text{otherwise}\end{cases}$$

and

$$\phi_3(x)=\begin{cases}1, & \text{if } x=1 \text{ or } 2\\ 0, & \text{otherwise.}\end{cases}$$

Then which of the following is/are true?

1. ϕ_1 is a most powerful test at level 0.05 for testing H_0 against H_1
2. ϕ_2 is a most powerful test at level 0.05 for testing H_0 against H_1
3. ϕ_3 is a most powerful test at level 0.05 for testing H_0 against H_1
4. ϕ_3 is unbiased at level 0.05 for testing H_0 against H_1

Answer Key December 2017 JRF NET EXAM

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