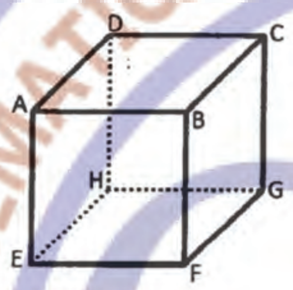
CSIR/UGC/NET MATHEMATICS DECEMBER 2012

1. Which of the following numbers is the largest?

$$2^{3^4}, 2^{4^3}, 3^{2^4}, 3^{4^2}, 4^{2^3}, 4^{3^2}$$

(a) 2^{3^4}
(b) 3^{4^2}
(c) 4^{3^2}
(d) 4^{2^3}

2. The cube ABCDEFGH in the figure has each edge equal to a. The area of the triangle with vertices A, C and F is



(a) $\frac{\sqrt{3}}{4}a^2$
(b) $\frac{\sqrt{3}}{2}a^2$
(c) $\sqrt{3}a^2$
(d) $2\sqrt{3}a^2$

3. What is the number of distinct arrangements of the letters of the word UGCCSIR so that U and I cannot come together?

(a) 2520
(b) 720
(c) 1520
(d) 1800

4. Suppose the sum of the seven positive numbers is 21. What is the minimum possible value of the average of the squares of these numbers?

(a) 63
(b) 21
(c) 9
(d) 7

5. Let

$$A = \frac{1^{13} + 2^{13} + 3^{13} + \cdots + 100^{13}}{100}, B = \frac{1^{13} + 5^{13} + \cdots + 99^{13}}{50}, C = \frac{2^{13} + 4^{13} + \cdots + 100^{13}}{50}.$$

Which of the following is true?

(a) $B < C < A$
(b) $A < B < C$
(c) $B < C < A$
(d) $C < A < B$

6. A circle of radius 5 units in the XY plane has its center in the first quadrant touches the X-axis and has a chord of length 6 units on the Y-axis. The coordinates of its center are

(a) $(4, 6)$
(b) $(3, 5)$
(c) $(5, 4)$
(d) $(4, 5)$

7. A wire of length 6 m is used to make a tetrahedron of each edge 1 m, using only one stand of wire for each edge. The minimum number of times the wire has to be cut is

(a) 2
(b) 3
(c) 1
(d) 0

8. If the sum of the next two terms of the series below is x, what is the value of $\log_e x$?

$$2, -4, 8, -16, 32, -64, 128 \cdots$$

(a) 128
(b) 10
(c) 256
(d) 8

9. A conical vessel with semi-vertical angle 30^o and height 10.5 cm has a thin lid. The radius of the sphere in cm is

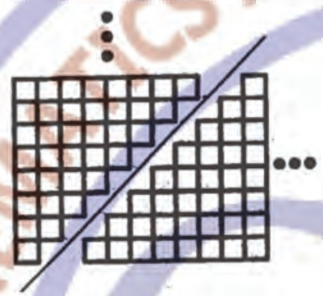
(a) 3.6
(b) 5
(c) 6.5
(d) 7

10. Amar, Akbar and Antony are 3 friends, one of whom is a doctor, another is an engineer and the third is a professor. Amar is not an engineer. Akbar is the shortest. The tallest person is a doctor. The engineer's height is the geometric mean of the heights of the other two. Then which of the following is true?

(a) Amar is a doctor and he is the tallest
(b) Akbar is a professor and he is the tallest
(c) Antony is a doctor and he is the shortest
(d) Antony is a doctor and he is the tallest

11. If 100 cats catch 100 mice in 100 minutes, then how long will take for 7 cats to catch 7 mice?
(a) 100/7 minutes
(b) 100 minutes
(c) 49/100 minutes
(d) 7 minutes

12. What does this diagram demonstrate?



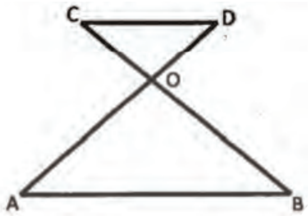
(a) $1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$
(b) $1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$
(c) $1 + 3 + 5 + \cdots + (2n-1) = n^2$
(d) $2^2 + 4^2 + 6^2 + \cdots + (2n)^2 = \frac{2n(n+1)(2n+1)}{3}$

13. Suppose there are socks of N different colors in a box. If you take out one sock at a time, what is the maximum number of socks that you have to take out before a matching pair is found? Assume that N is an even number.
(a) N
(b) $N + 1$
(c) $N - 1$
(d) $N/2$

14. At what time after 4 O' clock, the hour and the minute hands will lie opposite to each other?
(a) $4 - 50' - 31''$
(b) $4 - 52' - 51''$
(c) $4 - 53' - 23''$
(d) $4 - 54' - 33''$

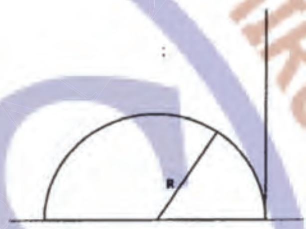
15. Which of the following curve just touches the X-axis?
(a) $y = x^2 - x + 1$
(b) $y = x^2 - 2x + 2$
(c) $y = x^2 - 10x + 25$
(d) $y = x^2 - 7x + 12$

16. If AB is parallel to CD and AO=2OD, then the area of the triangle OAB is bigger than the area of the triangle OCD by a factor of
(a) 2
(b) 3
(c) 4



(d) 8

17. A semi-circular arch of radius R has a vertical pole put on the the ground together with one of its legs. An ant on the top of the arch founds the angular height of the tip of the pole to be 45^o. The height of the pole is



(a) $\sqrt{2}R$
(b) $\sqrt{3}R$
(c) $\sqrt{4}R$
(d) $\sqrt{5}R$

18. Suppose we make N identical smaller spheres from a big sphere. The total surface area of the smaller spheres is X times the total surface area of the big sphere, where X is
(a) $\sqrt{N}$
(b) 1
(c) $N^{1/3}$
(d) N^3

19. What is the next number of the sequence

$$24, 30, 33, 39, 51, \cdots ?$$

(a) 57
(b) 69
(c) 54
(d) 81

20. Four lines are drawn on a plane with no two parallel and no three concurrent. Lines are drawn joining the points of intersection of the previous four lines. The number of new lines obtained in this way is
(a) 3
(b) 5
(c) 12
(d) 2

21. Let f be a twice differentiable function on $\mathbb{R}$. Given that $f''(x) > 0$ for all $x \in \mathbb{R}$,

1. $f(x) = 0$ has exactly two solutions on $\mathbb{R}$.
2. $f(x) = 0$ has a positive solution if $f(0) = 0$ and $f'(0) = 0$.
3. $f(x) = 0$ has no positive solution if $f(0) = 0$ and $f'(0) > 0$.
4. $f(x) = 0$ has no positive solution if $f(0) = 0$ and $f'(0) < 0$.

22. Let f be a continuously differentiable real-valued function on $[a,b]$ such that $|f'(x)| \le K$ for all $x \in [a,b]$. For a partition $P = \{a = a_0 < a_1 < \cdots < a_n = b\}$, let $U(f,P)$ and $L(f,P)$ denote the upper and lower Riemann sums of f with respect to P. Then

1. $|L(f,P)| \le K(b-a) \le |U(f,P)|$.
2. $U(f,P) - L(f,P) \le K(b-a)$.
3. $U(f,P) - L(f,P) \le K\|P\|$, where $\|P\| = \max_{0 \le i \le n-1}(a_{i+1} - a_i)$ is the norm of the partition.
4. $U(f,P) - L(f,P) \le K\|P\|(b-a)$.

23. Let f be a monotone nondecreasing real-valued function on $\mathbb{R}$. Then

1. $\lim_{x \to a} f(x)$ exists at each point a.
2. If $a < b$, then $\lim_{x \to a+} f(x) \le \lim_{x \to b-} f(x)$.
3. f is an unbounded function.
4. The function $g(x) = e^{-f(x)}$ is a bounded function.

24. Let f be a real-valued function on $\mathbb{R}^3$ satisfying (for a fixed $\alpha \in \mathbb{R}$) $f(rx) = r^\alpha f(x)$ for any $r > 0$ and $x \in \mathbb{R}^3$.

1. If $f(x) = f(y)$ whenever $\|x\| = \|y\| = \beta$ for a $\beta > 0$, then $f(x) = \beta\|x\|^\alpha$.
2. If $f(x) = f(y)$ whenever $\|x\| = \|y\| = 1$, then $f(x) = \|x\|^\alpha$.
3. If $f(x) = f(y)$ whenever $\|x\| = \|y\| = 1$, then $f(x) = c\|x\|^\alpha$, for some constant c.
4. If $f(x) = f(y)$ whenever $\|x\| = \|y\|$, then f must be a constant function.

25. Which of the following real-valued functions on (0,1) is uniformly continuous?

1. $f(x) = \frac{1}{x}$.
2. $f(x) = \frac{\sin x}{x}$.
3. $f(x) = \sin\frac{1}{x}$.
4. $f(x) = \frac{\cos x}{x}$.

26. Let X be a metric space and $A \subseteq X$ be a connected set with at least two distinct points. Then the number of distinct points in A is

1. 2.
2. more than 2, but finite.
3. countably infinite.
4. uncountable.

27. Let n be a positive integer and let $M_n(\mathbb{R})$ denote the space of all $n \times n$ real matrices. If $T : M_n(\mathbb{R}) \to M_n(\mathbb{R})$ is a linear transformation such that $T(A) = 0$ whenever $A \in M_n(\mathbb{R})$ is symmetric or skew-symmetric, then the rank of T is

1. $\frac{n(n+1)}{2}$.
2. $\frac{n(n-1)}{2}$.
3. n.
4. 0.

28. Let $S: \mathbb{R}^3 \to \mathbb{R}^4$ and $T: \mathbb{R}^4 \to \mathbb{R}^3$ be linear transformations such that $T \circ S$ is the identity map of $\mathbb{R}^3$. Then

1. $S \circ T$ is the identity map of $\mathbb{R}^4$.
2. $S \circ T$ is one-one, but not onto.
3. $S \circ T$ is onto, but not one-one.
4. $S \circ T$ is neither one-one nor onto.

29. Let V be a 3-dimensional vector space over the field $\mathbb{F}_3 = \mathbb{Z}/3\mathbb{Z}$ of 3 elements. The number of distinct 1-dimensional subspaces of V is

1. 13.
2. 26.
3. 9.
4. 15.

30. Let V be the inner product space consisting of linear polynomials, $p: [0, 1] \to \mathbb{R}$ (i.e., V consists of polynomials p of the form $p(x) = ax + b,\ a, b \in \mathbb{R}$), with the inner product defined by

$$\langle p, q \rangle = \int_0^1 p(x) q(x)\, dx \quad \text{for } p, q \in V.$$

An orthonormal basis of V is

1. $\{1, x\}$.
2. $\{1, x\sqrt{3}\}$.
3. $\{1, (2x-1)\sqrt{3}\}$.
4. $\{1, x - \frac{1}{2}\}$.

31. Let $f(x)$ be the minimal polynomial of the 4×4 matrix

$$A = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

Then the rank of the 4×4 matrix $f(A)$ is

1. 0. 2. 1. 3. 2. 4. 4.

32. Let a, b, c be positive real numbers such that $b^2 + c^2 < a < 1$. Consider the 3×3 matrix

$$A = \begin{bmatrix} 1 & b & c \\ b & a & 0 \\ c & 0 & 1 \end{bmatrix}.$$

1. All the eigenvalues of A are negative real numbers.
2. All the eigenvalues of A are positive real numbers.
3. A can have a positive as well as a negative eigenvalue.
4. Eigenvalues of A can be non-real complex numbers.

33. Consider the functions $f, g : \mathbb{C} \to \mathbb{C}$ defined by $f(z) = e^z$, $g(z) = e^{iz}$. Let $S = \{z \in \mathbb{C} : \operatorname{Re} z \in [-\pi, \pi]\}$. Then

1. f is an onto entire function.
2. g is a bounded function on $\mathbb{C}$.
3. f is bounded on S.
4. g is bounded on S.

34. Let $f : \mathbb{D} \to \mathbb{D}$ be a holomorphic function with $f(0) = 0$, where $\mathbb{D}$ is the open unit disc $\{z \in \mathbb{C}: |z| < 1\}$. Then

1. $|f'(0)| = 1$.
2. $|f(\frac{1}{2})| \le \frac{1}{2}$.
3. $|f(\frac{1}{2})| \le \frac{1}{4}$.
4. $|f'(0)| \le \frac{1}{2}$.

35. Consider the power series $\sum_{n=1}^{\infty} z^{n!}$. The radius of convergence of this series is

1. 0. 2. ∞. 3. 1. 4. a real number greater than 1.

36. In a group of 265 persons, 200 like singing, 110 like dancing and 55 like painting. If 60 persons like both singing and dancing, 30 like both singing and painting and 10 like all three activities, then the number of persons who like only dancing and painting is

1. 10. 2. 20. 3. 30. 4. 40.

37. The last two digits of 7^{81} are

1. 07. 2. 17. 3. 37. 4. 47.

38. In which of the following fields, the polynomial

$$x^3 - 312312x + 123123$$

is irreducible in $\mathbb{F}[x]$?

1. the field $\mathbb{F}_3$ with 3 elements.
2. the field $\mathbb{F}_7$ with 7 elements.
3. the field $\mathbb{F}_{13}$ with 13 elements.
4. the field $\mathbb{Q}$ of rational numbers.

39. Let ω be a complex number such that $\omega^3=1$ and $\omega \neq 1$. Suppose L is the field $\mathbb{Q}(\sqrt[3]{2},\omega)$ generated by $\sqrt[3]{2}$ and ω over the field $\mathbb{Q}$ of rational numbers. Then the number of subfields K of L such that $\mathbb{Q} \subsetneq K \subsetneq L$ is

1. 2. 2. 3. 3. 4. 4. 5.

40. Let $f: \mathbb{R}^n \to \mathbb{R}$ be a linear map with $f(0,\cdots,0)=0$. Then the set

$$\left\{ f(x_1, x_2, \cdots, x_n) : \sum_{j=1}^{n} x_j^2 \leq 1 \right\} \text{ equals}$$

1. $[-a, a]$ for some $a \in \mathbb{R}$, $a \geq 0$.
2. $[0, 1]$.
3. $[0, a]$ for some $a \in \mathbb{R}$, $a \geq 0$.
4. $[a, b]$ for some $a, b \in \mathbb{R}$, $0 \leq a < b$.

41. Let $y_1(x)$ and $y_2(x)$ be the solutions of the differential equation $\frac{dy}{dx} = y + 17$ with initial conditions $y_1(0)=0$, $y_2(0)=1$. Then

1. y_1 and y_2 will never intersect.
2. y_1 and y_2 will intersect at $x=17$.
3. y_1 and y_2 will intersect at $x=e$.
4. y_1 and y_2 will intersect at $x=1$.

42. Let $y(t)=\begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix}$ satisfy

$$\frac{dy}{dt}=Ay;\ t>0$$
$$y(0)=\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

where A is a 2×2 constant matrix with real entries satisfying trace $A=0$ and det $A>0$. Then $y_1(t)$ and $y_2(t)$ both are

1. monotonically decreasing functions of t.
2. monotonically increasing functions of t.
3. oscillating functions of t.
4. constant functions of t.

43. The partial differential equation

$$y\frac{\partial^2 u}{\partial x^2}+x\frac{\partial^2 u}{\partial y^2}=0$$

is hyperbolic in

1. the second and fourth quadrants.
2. the first and second quadrants.
3. the second and third quadrants.
4. the first and third quadrants.

44. A bounded harmonic function in the unit disc centered at origin and taking the value $\sin 2\theta$ on the boundary is

1. $r^2\sin 2\theta$.
2. $r\sin 2\theta$.
3. $\frac{1}{r}\sin 2\theta$.
4. $\frac{1}{r^2}\sin 2\theta$.

45. The system of equations

$$x+y+z=1$$
$$2x+3y-z=5$$
$$x+2y-kz=4$$

where $k\in\mathbb{R}$, has an infinite number of solutions for

1. $k=0$.
2. $k=1$.
3. $k=2$.
4. $k=3$.

46. Let

$$J(u)=\int_0^1\left[u_x^2+4\frac{u^2}{x^2}\right]xdx,$$

where $u(x)$ is a smooth function defined on $[0,1]$ satisfying $u(0)=0$ and $u(1)=1$. Which of the functions minimizes J?

1. $u(x)=x^2$.
2. $u(x)=\frac{1}{\sqrt{2}}x^2$.
3. $u(x)=\frac{1}{2}x^2$.
4. $u(x)=\frac{1}{4}x^2$.

47. For the homogeneous Fredholm integral equation $\phi(x)=\lambda\int_0^1 e^{x+t}\phi(t)\,dt$, a non-trivial solution exists, when λ has the value

1. $\lambda=\frac{2}{e-1}$.
2. $\lambda=\frac{1}{e^2+1}$.
3. $\lambda=\frac{1}{e+1}$.
4. $\lambda=\frac{2}{e^2-1}$.

48. The total number of vibrational degrees of freedom of a molecule containing n collinear atoms is

1. $3n-6$.
2. $3n-5$.
3. $3n-2$.
4. $3n$.

49. Let $X_1, X_2, \cdots$ be i.i.d. standard normal random variables and let $T_n=\frac{X_1^2+\cdots+X_n^2}{n}$. Then

1. The limiting distribution of T_n-1 is χ^2 with 1 degree of freedom.
2. The limiting distribution of $\frac{T_n-1}{\sqrt{n}}$ is normal with mean 0 and variance 2.
3. The limiting distribution of $\sqrt{n}(T_n-1)$ is χ^2 with 1 degree of freedom.
4. The limiting distribution of $\sqrt{n}(T_n-1)$ is normal with mean 0 and variance 2.

50. Let $\{p_n, n\ge 0\}$ be a sequence of numbers, such that for all $n\ge 0$, $p_n>0$, $\sum_{n=0}^{\infty}p_n=1$ and $\sum_{n=0}^{\infty}np_n<\infty$. Consider a Markov chain on the state space $\{0,1,2,\cdots\}$ with transition probability matrix

$$\begin{pmatrix} p_0 & p_1 & p_2 & \cdots \\ 1 & 0 & 0 & \cdots \\ 0 & 1 & 0 & \cdots \\ 0 & 0 & 1 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Then

1. the chain is not irreducible.
2. the chain is irreducible and transient.
3. the chain is irreducible and null recurrent.
4. the chain is irreducible and positive recurrent.

51. Suppose X_1, X_2, X_3, X_4 are i.i.d. random variables taking values 1 and -1 with probability ½ each. Then $E(X_1+X_2+X_3+X_4)^4$ equals

1. 4.
2. 76.
3. 16.
4. 12.

52. Suppose that $\begin{pmatrix} X_1 \\ X_2 \end{pmatrix}$ has Normal $(\mu_{2\times1}, \Sigma_{2\times2})$ distribution, where $\Sigma_{2\times2}$ is nonsingular. Let $X_3 = -2X_2$. Then which of the following has a singular normal distribution?

1. $\begin{pmatrix} X_1 - 2X_2 \\ X_2 - 2X_3 \end{pmatrix}$.
2. $\begin{pmatrix} X_1 - X_2 - X_3 \\ 2X_1 + 2X_2 \end{pmatrix}$.
3. $\begin{pmatrix} X_1 + X_2 \\ 2X_1 + 2X_3 \end{pmatrix}$.
4. $\begin{pmatrix} X_1 + X_2 + X_3 \\ X_1 + X_2 \end{pmatrix}$.

53. The radius of a circle is measured with an error of measurement which is normally distributed with mean 0 and variance σ^2. Let $X_1, \cdots, X_n$ be n measurements on the radius. Let $\bar{x}$ and $s^2 = \frac{1}{n-1}\Sigma(x_i - \bar{x})^2$ be the sample mean and sample variance respectively. Which of the following represents an unbiased estimate of the area of the circle?

54. Suppose person A and person B draw random samples of sizes 15 and 20 respectively from $N(\mu,\sigma^2)$, $\sigma^2>0$ for testing H_o: $\mu=2$ against H_1: $\mu>2$. In both the cases, the observed sample means and sample standard deviations are same with values $\bar{x}_1 = \bar{x}_2 = 1.8$, $s_1 = s_2 = s$. Both of them use the usual t-test & state the p-values p_A and p_B respectively. Then which of the following is correct?

1. $p_A > p_B$.
2. $p_A = p_B$.
3. $p_A < p_B$.
4. Relation between p_A and p_B depends on the value of s.

55. In logistic regression model involving a binary response Y and two explanatory variables X_1 and X_2, the coefficient of X_2 is

1. $\dfrac{P(Y=1|X_1, X_2+1)}{P(Y=1|X_1, X_2)}$.
2. $\log\left\{\dfrac{P(Y=1|X_1, X_2+1)}{P(Y=1|X_1, X_2)}\right\}$.
3. $\dfrac{P(Y=1|X_1, X_2+1)}{P(Y=0|X_1, X_2+1)} \times \dfrac{P(Y=0|X_1, X_2)}{P(Y=1|X_1, X_2)}$.
4. $\log\left\{\dfrac{P(Y=1|X_1, X_2+1)}{P(Y=0|X_1, X_2+1)} \times \dfrac{P(Y=0|X_1, X_2)}{P(Y=1|X_1, X_2)}\right\}$.

56. The failure rate of a parallel system of two components, where the component lifetimes are independent and have the exponential distribution with mean 2, is

1. a constant.
2. a monotone and bounded function.
3. a monotone and unbounded function.
4. a non-monotone function.

57. Consider the linear model

$$y_1 = \theta_1 + 2\theta_2 - 2\theta_3 + \varepsilon_1$$

$$y_2 = \theta_1 + 3\theta_2 - \theta_3 + \varepsilon_2$$

$$y_3 = \theta_2 + \theta_3 + \varepsilon_3,$$

where y_i are observations, θ_i are parameters and ε_i are uncorrelated random variables with mean zero and constant variance for i=1, 2, 3. Then which of the following is true?

1. $2y_1 - y_2 - y_3$ is an unbiased estimator of $\theta_1 - 4\theta_3$.
2. $2y_1 - y_2 - y_3$ is the BLUE of $\theta_1 - 4\theta_3$.
3. $y_2 - 3y_3$ is the BLUE of $\theta_1 - 4\theta_3$.
4. $y_1 - 4y_3$ is an unbiased estimator of $\theta_1 - 4\theta_3$.

58. Consider the following design with 6 treatments 1, 2,…,6 and 4 blocks as follows: {1,2,2}, {2,3,3}, {3,4,4} and {5,6,6}. Which of the following is true?

1. The design is orthogonal and all treatment contrasts are estimable.
2. The design is non-orthogonal and all treatment contrasts are estimable.
3. The design is orthogonal and not all treatment contrasts are estimable.
4. The design is non-orthogonal and not all treatment contrasts are estimable.

59. Consider a population of 50 units {1,2,…,50} and suppose that 50 possible samples are listed as : {1}, {1,2}, {1,2,3}, {1,2,3,4}, …, {1,2,3,…,50}. One of these samples is chosen at random. Let π_i be the probability that unit i is in the selected sample. Then which of the following is necessarily true?

1. The expected sample size is 25.
2. The expected sample size is 25.5.
3. $\sum_{i=1}^{50} \pi_i = 1$.
4. $\sum_{i=1}^{50} \pi_i = 25$.

60. Men arrive in a queue according to a Poisson process with rate λ_1 and women arrive in the same queue according to another Poisson process with rate λ_2. The arrivals of men and women are independent. The probability that the first arrival in the queue is a man is

1. $\frac{\lambda_1}{\lambda_1 + \lambda_2}$.
2. $\frac{\lambda_2}{\lambda_1 + \lambda_2}$.
3. $\frac{\lambda_1}{\lambda_2}$.
4. $\frac{\lambda_2}{\lambda_1}$.

(Part C)

(Unit – 1)

61. Let $\{f_n\}$ be a sequence of continuous real-valued functions defined on $[0,\infty)$. Suppose $f_n(x) \to f(x)$ for all $x \in [0,\infty)$ and that f is integrable. Then

1. $\int_0^\infty f_n(x)\,dx \to \int_0^\infty f(x)\,dx$ as $n \to \infty$.
2. if $f_n \to f$ uniformly on $[0,\infty)$, then $\int_0^1 f_n(x)\,dx \to \int_0^1 f(x)\,dx$.
3. if $f_n \to f$ uniformly on $[0,\infty)$, then $\int_0^\infty f_n(x)\,dx \to \int_0^\infty f(x)\,dx$.
4. if $\int_0^1 |f_n(x) - f(x)|\,dx \to 0$, then $f_n \to f$ uniformly on [0, 1].

62. Let X be a compact topological space and let $f: X \to \mathbb{R}$ be a function. The graph of f is the set $G = \{(x, f(x)) : x \in X\} \subseteq X \times \mathbb{R}$. Which of the following are necessarily true?

1. G is a closed set if and only if f is continuous.
2. If f is continuous, then G is closed.
3. If f is continuous, then G is connected.
4. If f is a bounded continuous function, then G is compact.

63. Suppose $f: \mathbb{R} \to \mathbb{R}$ is a differentiable function. Then which of the following statements are necessarily true?

1. If $f'(x) \le r < 1$ for all $x \in \mathbb{R}$, then f has at least one fixed point.
2. If f has a unique fixed point, then $f'(x) \le r < 1$ for all $x \in \mathbb{R}$.
3. If f has a unique fixed point, then $f'(x) \ge r > -1$ for all $x \in \mathbb{R}$.
4. If $f'(x) \le r < 1$ for all $x \in \mathbb{R}$, then f has a unique fixed point.

64. Which of the conditions below imply that a function $f:[0,1] \to \mathbb{R}$ is necessarily of bounded variation?

1. f is a monotone function on [0, 1].
2. f is a continuous and monotone function on [0, 1].
3. f has a derivative at each $x \in (0,1)$.
4. f has a bounded derivative on the interval (0, 1).

65. Let $f(x) = \sin x - x + \frac{x^3}{3!}$ and $g(x) = \cos x - 1 + \frac{x^2}{2!}$ for $x \in \mathbb{R}$. Which of the following statements are correct?

1. $f(x) \ge 0$ for all $x > 0$.
2. g is an increasing function on $[0,\infty)$.
3. g is a decreasing function on $[0,\infty)$.
4. f is a decreasing function on $[0,\infty)$.

66. Let $f: A \cup E \to \mathbb{R}^2$ be differentiable, where $A = \left\{(x,y) \in \mathbb{R}^2 : \frac{1}{2} < x^2 + y^2 < 1\right\}$ and $E = \left\{(x,y) \in \mathbb{R}^2 : (x-2)^2 + (y-2)^2 < \frac{1}{2}\right\}$. Let Df be the derivative of the function f. Which of the following are necessarily correct?

1. If $(Df)(x,y) = 0$ for all $(x,y) \in A \cup E$, then f is constant.
2. If $(Df)(x,y) = 0$ for all $(x,y) \in A$, then f is constant on A.
3. If $(Df)(x,y) = 0$ for all $(x,y) \in E$, then f is constant on E.
4. If $(Df)(x,y) = 0$ for all $(x,y) \in A \cup E$, then, for some $(x_0,y_0),(x_1,y_1) \in \mathbb{R}^2$, $f(x,y) = (x_0,y_0)$ for all $(x,y) \in A$ and $f(x,y) = (x_1,y_1)$ for all $(x,y) \in E$.

67. Let $L: \mathbb{R}^n \to \mathbb{R}$ be the function $L(x) = \langle x, y \rangle$, where $\langle \cdot, \cdot \rangle$ is some inner product on $\mathbb{R}^n$ and y is a fixed vector in $\mathbb{R}^n$. Further denote by DL, the derivative of L. Which of the following are necessarily correct?

1. $DL(u) = DL(v)$ for all $u, v \in \mathbb{R}^n$.
2. $DL(0,0,\cdots,0) = L$.
3. $DL(x) = \|x\|^2$ for all $x \in \mathbb{R}^n$.
4. $DL(1,1,\cdots,1) = 0$.

68. Let $f: [\pi, 2\pi] \to \mathbb{R}^2$ be the function $f(t) = (\cos t, \sin t)$. Which of the following are necessarily correct?

1. There exists $t_0 \in [\pi, 2\pi]$ such that $f'(t_0) = \frac{1}{\pi}\left(f(2\pi) - f(\pi)\right)$.
2. There does not exist any $t_0 \in [\pi, 2\pi]$ such that $f'(t_0) = \frac{1}{\pi}\left(f(2\pi) - f(\pi)\right)$
3. There exists $t_0 \in [\pi, 2\pi]$ such that $\| f(2\pi) - f(\pi) \| \le \pi \| f'(t_0) \|$.
4. $f'(t) = (-\sin t, \cos t)$ for all $t \in [\pi, 2\pi]$.

69. Let $X = [-1,1] \times [-1,1]$, $A = \{(x,y) \in X : x^2 + y^2 = 1\}$, $B = \{(x,y) \in X : |x| + |y| = 1\}$, $C = \{(x,y) \in X : xy = 0\}$ and $D = \{(x,y) \in X : x = \pm y\}$. Then

1. A is homeomorphic to B.
2. B is homeomorphic to C.
3. C is homeomorphic to D.
4. D is homeomorphic to A.

70. Let n be an integer, $n \ge 3$, and let $u_1, u_2, \ldots, u_n$ be n linearly independent elements in a vector space over $\mathbb{R}$. Set $u_0 = 0$ and $u_{n+1} = u_1$. Define $v_i = u_i + u_{i+1}$ and $w_i = u_{i-1} + u_i$ for $i = 1, 2, \cdots, n$. Then

1. $v_1, v_2, \cdots, v_n$ are linearly independent, if $n = 2010$.
2. $v_1, v_2, \cdots, v_n$ are linearly independent, if $n = 2011$.
3. $w_1, w_2, \cdots, w_n$ are linearly independent, if $n = 2010$.
4. $w_1, w_2, \cdots, w_n$ are linearly independent, if $n = 2011$.

71. Let V and W be finite-dimensional vector spaces over $\mathbb{R}$ and let $T_1: V \to V$ and $T_2: W \to W$ be linear transformations whose minimal polynomials are given by

$$f_1(x) = x^3 + x^2 + x + 1 \quad \text{and} \quad f_2(x) = x^4 - x^2 - 2.$$

Let $T: V \oplus W \to V \oplus W$ be the linear transformation defined by

$$T(v, w) = (T_1(v), T_2(w)) \quad \text{for } (v, w) \in V \oplus W$$

and let $f(x)$ be the minimal polynomial of T. Then

1. $\deg f(x) = 7$.
2. $\deg f(x) = 5$.
3. nullity $(T) = 1$.
4. nullity $(T) = 0$.

72. Let $a,b,c,d \in \mathbb{R}$ and let $T:\mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation defined by

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} ax+by \\ cx+dy \end{bmatrix} \quad \text{for } \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2,$$

Let $S:\mathbb{C} \to \mathbb{C}$ be the corresponding map defined by

$$S(x+iy) = (ax+by) + i(cx+dy) \quad \text{for } x, y \in \mathbb{R}.$$

Then

1. S is always $\mathbb{C}$-linear, that is $S(z_1 + z_2) = S(z_1) + S(z_2)$ for all $z_1, z_2 \in \mathbb{C}$ and $S(\alpha z) = \alpha S(z)$ for all $\alpha \in \mathbb{C}$ and $z \in \mathbb{C}$.
2. S is $\mathbb{C}$-linear if $b = -c$ and $d = a$.
3. S is $\mathbb{C}$-linear only if $b = -c$ and $d = a$.
4. S is $\mathbb{C}$-linear if and only if T is the identity transformation.

73. Let $A = (a_{ij})$ be an $n \times n$ complex matrix and let A^* denote the conjugate transpose of A. Which of the following statements are necessarily true?

1. If A is invertible, then $\mathrm{tr}(A^*A) \neq 0$, i.e., the trace of A^*A is nonzero.
2. If $\mathrm{tr}(A^*A) \neq 0$, then A is invertible.
3. If $|\mathrm{tr}(A^*A)| < n^2$, then $|a_{ij}| < 1$ for some i, j.
4. If $\mathrm{tr}(A^*A) = 0$, then A is the zero matrix.

74. Let n be a positive integer and V be an $(n + 1)$ - dimensional vector space over $\mathbb{R}$. If $\{e_1, e_2, \ldots, e_{n+1}\}$ is a basis of V and $T: V \to V$ is the linear transformation satisfying

$$T(e_i) = e_{i+1} \text{ for } i = 1, 2, \ldots, n \text{ and } T(e_{n+1}) = 0.$$

Then

1. trace of T is nonzero.
2. rank of T is n.
3. nullity of T is 1.
4. $T^n = T \circ T \circ \cdots \circ T$ (n times) is the zero map.

75. Let A and B be $n \times n$ real matrices such that $AB = BA = 0$ and $A+B$ is invertible. Which of the following are always true?

1. rank (A) = rank (B).
2. rank (A) + rank (B) = n.
3. nullity (A) + nullity (B) = n.
4. $A-B$ is invertible.

76. Let n be an integer ≥ 2 and let $M_n(\mathbb{R})$ denote the vector space of $n \times n$ real matrices. Let $B \in M_n(\mathbb{R})$ be an orthogonal matrix and let B^t denote the transpose of B. Consider $W_B = \{B^t AB : A \in M_n(\mathbb{R})\}$. Which of the following are necessarily true?

1. W_B is a subspace of $M_n(\mathbb{R})$ and dim $W_B \leq$ rank (B).
2. W_B is a subspace of $M_n(\mathbb{R})$ and dim $W_B =$ rank (B) rank (B^t).
3. $W_B = M_n(\mathbb{R})$.
4. W_B is not a subspace of $M_n(\mathbb{R})$.

77. Let A be a 5×5 skew-symmetric matrix with entries in $\mathbb{R}$ and B be the 5×5 symmetric matrix whose $(i, j)^{th}$ entry is the binomial coefficient $\binom{i}{j}$ for $1 \leq i \leq j \leq 5$. Consider the 10×10 matrix, given in block form by

$$C = \begin{pmatrix} A & A+B \\ 0 & B \end{pmatrix}.$$

Then

1. det $C = 1$ or -1.
2. det $C = 0$.
3. trace of C is 0.
4. trace of C is 5.

78. Suppose A is a 3×3 symmetric matrix such that

$$[x, y, 1] A \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = xy - 1.$$

Let p be the number of positive eigenvalues of A and let $q =$ rank $(A) - p$. Then

1. $p = 1$.
2. $p = 2$.
3. $q = 2$.
4. $q = 1$.

(Unit -11)

79. Which of the following functions f are entire functions and have simple zeros at $z = ik$ for all $k \in \mathbb{Z}$.

1. $f(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_0$ for some $n \geq 1$ and some $a_0, a_1, \cdots, a_n \in \mathbb{C}$.
2. $f(z) = a \sin 2\pi i z$, for some $a \in \mathbb{C}$.
3. $f(z) = b \cos 2\pi (iz - 1/4)$, for some $b \in \mathbb{C}$.
4. $f(z) = e^{cz}$, for some $c \in \mathbb{C}$.

80. Let $\gamma_k = \{ke^{ik\theta} : 0 \le \theta \le 2\pi\}$ for $k = 1,2,3$. Which of the following are necessarily correct?

1. $\frac{1}{2\pi i}\int_{\gamma_k}\frac{1}{z}dz = 0$ for $k = 1,2,3$.
2. $\frac{1}{2\pi i}\int_{\gamma_1}\frac{1}{z}dz = 1$.
3. $\frac{1}{2\pi i}\int_{\gamma_2}\frac{1}{z}dz = 4$.
4. $\frac{1}{2\pi i}\int_{\gamma_3}\frac{1}{z}dz = 3$.

81. Let f be an analytic function defined on $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ such that the range of f is contained in the set $\mathbb{C} \setminus (-\infty, 0]$. Then

1. f is necessarily a constant function.
2. there exists an analytic function g on $\mathbb{D}$ such that $g(z)$ is a square root of $f(z)$ for each $z \in \mathbb{D}$.
3. there exists an analytic function g on $\mathbb{D}$ such that $\text{Re } g(z) \ge 0$ and $g(z)$ is a square root of $f(z)$ for each $z \in \mathbb{D}$.
4. there exists an analytic function g on $\mathbb{D}$ such that $\text{Re } g(z) \le 0$ and $g(z)$ is a square root of $f(z)$ for each $z \in \mathbb{D}$.

82. Let $f : \Omega \to \mathbb{C}$ be an analytic function on an open set $\Omega \subseteq \mathbb{C}$. For $r > 0$, let $\mathbb{D}_r = \{z \in \mathbb{C} : |z| < r\}$ and let $\overline{\mathbb{D}_r}$ be its closure. Which of the following are necessarily true?

1. If $\overline{\mathbb{D}_1} \subset f(\Omega)$, then $\mathbb{D}_r \subset f(\Omega)$ for some $r > 1$.
2. If $\overline{\mathbb{D}_1} \subset f(\Omega)$, then $\mathbb{D}_r = f(\Omega)$ for some $r > 1$.
3. If $\overline{\mathbb{D}_1} \subset f(\Omega)$, then $\overline{\mathbb{D}_r} \subset f(\Omega)$ for some $r > 1$.
4. $f(\Omega)$ is open.

83. Let $f(z) = z + \frac{1}{z}$ for $z \in \mathbb{C}$ with $z \ne 0$. Which of the following are always true?

1. f is an analytic function on $\mathbb{C} \setminus \{0\}$.
2. f is a conformal map on $\mathbb{C} \setminus \{0\}$.
3. f maps the unit circle to a subset of the real axis.
4. The image of any circle in $\mathbb{C} \setminus \{0\}$ is again a circle.

84. For a positive integer m, let $\varphi(m)$ denote the number of integers k such that $1 \le k \le m$ and $\text{GCD}(k, m) = 1$. Then which of the following statements are necessarily true?

1. $\varphi(n)$ divides n for every positive integer n.
2. n divides $\varphi(a^n - 1)$ for all positive integers a and n.
3. n divides $\varphi(a^n - 1)$ for all positive integers a and n such that $\text{GCD}(a, n) = 1$.
4. a divides $\varphi(a^n - 1)$ for all positive integers a and n such that $\text{GCD}(a, n) = 1$.

85. For a positive integer $n \geq 4$ and a prime number $p \leq n$, let $U_{p,n}$ denote the union of all p-Sylow subgroups of the alternating group A_n on n letters. Also let $K_{p,n}$ denote the subgroup of A_n generated by $U_{p,n}$, and let $|K_{p,n}|$ denote the order of $K_{p,n}$. Then

1. $|K_{2,4}| = 12$.
2. $|K_{2,4}| = 4$.
3. $|K_{2,5}| = 60$.
4. $|K_{3,5}| = 30$.

86. For a positive integer n, let $f_n(x) = x^{n-1} + x^{n-2} + \cdots + x + 1$. Then

1. $f_n(x)$ is an irreducible polynomial in $\mathbb{Q}[x]$ for every positive integer n.
2. $f_p(x)$ is an irreducible polynomial in $\mathbb{Q}[x]$ for every prime number p.
3. $f_{p^e}(x)$ is an irreducible polynomial in $\mathbb{Q}[x]$ for every prime number p and every positive integer e.
4. $f_p\left(x^{p^{e-1}}\right)$ is an irreducible polynomial in $\mathbb{Q}[x]$ for every prime number p and every positive integer e.

87. Consider the ring $R = \mathbb{Z}\left[\sqrt{-5}\right] = \left\{a + b\sqrt{-5} : a, b \in \mathbb{Z}\right\}$ and the element $\alpha = 3 + \sqrt{-5}$ of R. Then

1. α is prime.
2. α is irreducible.
3. R is not a unique factorization domain.
4. R is not an integral domain.

88. Consider the polynomial $f(x) = x^4 - x^3 + 14x^2 + 5x + 16$. Also for a prime number p, let $\mathbb{F}_p$ denote the field with p elements. Which of the following are always true?

1. Considering f as a polynomial with coefficients in $\mathbb{F}_3$, it has no roots in $\mathbb{F}_3$.
2. Considering f as a polynomial with coefficients in $\mathbb{F}_3$, it is a product of two irreducible factors of degree 2 over $\mathbb{F}_3$.
3. Considering f as a polynomial with coefficients in $\mathbb{F}_7$, it has an irreducible factor of degree 3 over $\mathbb{F}_7$.
4. f is a product of two polynomials of degree 2 over $\mathbb{Z}$.

89. For a positive integer m, let a_m denote the number of distinct prime ideals of the ring $\dfrac{\mathbb{Q}[x]}{\langle x^m - 1\rangle}$.

Then

1. $a_4 = 2$.
2. $a_4 = 3$.
3. $a_5 = 2$.
4. $a_5 = 3$.

90. Let τ be the topology on $\mathbb{R}$ for which the intervals $[a,b)$, $-\infty < a < b < \infty$, form a base. Let σ be a topology on $\mathbb{R}$ such that $\sigma \supseteq \tau$. Then

1. either $\sigma = \tau$ or σ is the discrete topology.
2. if, moreover, the map $x \mapsto -x$ is continuous for σ, then σ is the discrete topology.
3. if, moreover, the map $x \mapsto -x$ is a homeomorphism for σ, then σ is the discrete topology.
4. if, moreover, the map $x \mapsto |x|$ is a homeomorphism for σ, then σ is the discrete topology.

(Unit - 111)

91. The differential equation

$$\frac{dy}{dx}=60(y^2)^{1/5};\quad x>0$$
$$y(0)=0$$

has

1. a unique solution.
2. two solutions.
3. no solution.
4. infinite number of solutions.

92. The solution of the differential equation

$$\frac{d^2y}{dx^2}=f(x);\quad x\in(0,1)$$
$$y(0)=y(1)=0$$

is given by

$$y(x)=\int_0^1 G(x,\xi)f(\xi)\,d\xi$$

where

1. $G(x,\xi)=\begin{cases} x(\xi-1); & x\le\xi \\ \xi(x-1); & x>\xi \end{cases}$.
2. $G(x,\xi)=\begin{cases} x^2(\xi-1); & x\le\xi \\ \xi^2(x-1); & x>\xi \end{cases}$.
3. $G(x,\xi)=\begin{cases} x(\xi^2-1); & x\le\xi \\ \xi(x^2-1); & x>\xi \end{cases}$.
4. $G(x,\xi)=\begin{cases} \sin x(\xi-1); & x\le\xi \\ \sin\xi(x-1); & x>\xi \end{cases}$.

93. A bounded solution to the partial differential equation

$$\frac{\partial u}{\partial t}=\frac{\partial^2 u}{\partial x^2}+e^{-t}$$

is

1. $u(x,t)=-e^{-t}$.
2. $u(x,t)=e^{-x}e^{-t}$.
3. $u(x,t)=e^{-x}+e^{-t}$.
4. $u(x,t)=x-e^{-t}$.

94. If $u(x,t)$ satisfy the partial differential equation

$$\frac{\partial^2 u}{\partial t^2}=4\frac{\partial^2 u}{\partial x^2}$$

then $u(x,t)$ can be of the form

1. $u(x,t)=f(e^{x-2t})+g(x+2t)$.
2. $u(x,t)=f(x^2-4t^2)+g(x^2+4t^2)$.
3. $u(x,t)=f(2x-4t)+g(x+2t)$.
4. $u(x,t)=f(2x-t)+g(2x+t)$.

95. Let f be a continuous map from the interval $[0,1]$ into itself and consider the iteration

$$x_{n+1} = f(x_n).$$

Which of the following maps will yield a fixed point for f ?

1. $f(x)=x^2/4$.
2. $f(x)=x^2/8$.
3. $f(x)=x^2/16$.
4. $f(x)=x^2/32$.

96. Consider the ordinary differential equation

$$\frac{dy}{dt}=\lambda y;\ t>0$$
$$y(0)=1,$$

and the Euler scheme with step size h

$$\frac{Y_{n+1}-Y_n}{h}=\lambda Y_n;\ n\geq 1$$
$$Y_0=1$$

Which of the following are necessarily true for Y_1 which approximates $Y(h)=e^{\lambda h}$?

1. Y_1 is a polynomial approximation.
2. Y_1 is a rational function approximation.
3. Y_1 is a trigonometric function approximation.
4. Y_1 is a truncation of infinite series.

97. Consider the functional

$$J=\int_a^b F(x,y,y')\,dx$$

where $F(x,y,y')=(1+y^2)/y'^2$

for admissible functions $y(x)$. Which of the following are extremals for J?

1. $y(x) = A\sin(x)$.
2. $y(x) = A\sinh(x) + B\cosh(x)$.
3. $y(x) = A\sinh(Ax + B)$.
4. $y(x) = A\sin(x) + B\cos(x)$.

98. The initial value problem

$$\frac{d^2y}{dx^2}+y=0;\ x>0$$
$$y(0)=1,$$
$$y'(0)=0,$$

is equivalent to the Volterra integral equation

1. $y(x)=1+\int_0^x (t-x)\,y(t)\,dt$.
2. $y(x)=1+\int_0^x (t+x)\,y(t)\,dt$.
3. $y(x)=1+\int_0^x xty(t)\,dt$.
4. $y(x)=1+\int_0^x (x-t)\,y(t)\,dt$.

99. The integral equation

$$\varphi(x)-\lambda\int_{-1}^{1}\cos\left[\pi(x-t)\right]\varphi(t)\,dt=f(x)$$

has

1. a unique solution for $\lambda\neq 1$ when $f(x)=x$.
2. no solution for $\lambda\neq 1$ when $f(x)=1$.
3. no solution for $\lambda=1$ when $f(x)=x$.
4. infinite number of solutions for $\lambda=1$ when $f(x)=1$.

100. Let $f(u)=u^3-u-1$.

1. Starting with the initial guess $u^{(0)}=1.5$, the fixed point iterates of the equation $u=g(u)$, where $g(u)=u^3-1$ converge.
2. Starting with the initial guess $u^{(0)}=1.5$, the fixed point iterates of the equation $u=\tilde{g}(u)$, where $\tilde{g}(u)=\sqrt{1+u^3}$ converge.
3. If u^* is a root of the equation $f(u)=0$ and $u^*>1$, then u^* is a stable fixed point of the equation $u=g(u)$.
4. $f(u)=0$ has a root between 1 and 2.

101. To compute the value of e^t in the interval [0,1], pick $t_1=0, t_2=0.5$ and $t_3=1$. Let p be the quadratic polynomial that interpolates e^t, that is, $p(t_i)=e^{t_i}$, $i=1,2,3$. Then

1. the polynomial p can be written in the form $L_1(t)+e^{1/2}L_2(t)+eL_3(t)$ for some choice of quadratic polynomials L_1,L_2,L_3.
2. if the polynomial p is written in the form $L_1(t)+e^{1/2}L_2(t)+eL_3(t)$, where L_1,L_2 and L_3 are polynomials, then L_1,L_2, and L_3 are uniquely determined.
3. if p is written in the form $L_1(t)+e^{1/2}L_2(t)+eL_3(t)$, then one of L_1,L_2, or L_3 must be linear.
4. the polynomial p is uniquely determined.

102. The energy of a pendulum executing small oscillations in a Cartesian coordinate system is proportional to

1. square of the amplitude.
2. square of the frequency .
3. its mass.
4. inverse of the product of mass, frequency and amplitude.

(Unit - 111)

103. Let $\{X_n : n \geq 0\}$ and X be random variables defined on a common probability space. Further assume that X_n's are nonnegative and X takes values 0 and 1 with probability p and $1-p$ respectively, where $0 \leq p \leq 1$. Which of the following statements are necessarily true?

1. If $p = 0$ and X_n converges to X in distribution, then X_n converges to X in probability.
2. If $p = 1$ and X_n converges to X in distribution, then X_n converges to X in probability.
3. If $0 < p < 1$ and X_n converges to X in distribution, then X_n converges to X in probability.
4. If X_n converges to X in probability, then X_n converges to X almost surely.

104. Let X be a binomial random variable with parameters $\left(11, \frac{1}{3}\right)$. At which value(s) of k is $P(X=k)$ maximized?

1. $k=2$.
2. $k=3$.
3. $k=4$.
4. $k=5$.

105. X, Y, Z are independent random variables with N(0,1) (standard normal) distribution. Let $f:\mathbb{R} \to \mathbb{R}$ be defined by $f(x)=1$, if $x \geq 0$ and $f(x)=-1$, if $x<0$. Let U, V, W be defined by $U=|X| \cdot f(Y)$, $V=|Y| \cdot f(X)$, $W=|Z| \cdot f(X)$. Then

1. U and V are independent each having a N(0,1) distribution.
2. U and W are independent each having N(0,1) distribution.
3. V and W are independent each having N(0,1) distribution.
4. U, V and W are independent random variables.

106. One chooses letters with replacement from the set $\{a, b, c, d\}$ as follows:

After one letter is observed at step n, in $(n+1)^{th}$ step another letter will be chosen from the other three with equal probability 1/3. Let Xn denote the letter chosen at the nth step. Which of the following are necessarily true for the Markov chain X_n?

1. $P[X_n = a]$ converges to 1/3.
2. $P[X_n = b]$ converges to 1/4.
3. The average proportion of times c is observed converges to $\frac{1}{4}$.
4. The chain is <u>not</u> irreducible.

107. Consider a Markov chain $\{X_n : n \geq 0\}$ on the state space $\{0, 1\}$ with transition probability matrix P. Which of the following statements are necessarily true?

1. When $P=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$, $\lim_{n\to\infty} P_\nu[X_n=i]$ converges for $i = 0, 1$, but the limits depend on the initial distribution ν.
2. When $P=\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$, $\lim_{n\to\infty} P_\nu[X_n=1]$ exists and is positive for all choices of the initial distribution ν.
3. When $P=\begin{pmatrix}1/2 & 1/2\\ 0 & 1\end{pmatrix}$, $\lim_{n\to\infty} P_\nu[X_n=0]$ does not exist for any choice of the initial distribution ν.
4. When $P=\begin{pmatrix}1/2 & 1/2\\ 1 & 0\end{pmatrix}$, $\lim_{n\to\infty} P_\nu[X_n=0]$ always exists, but may be 0 for some choice of the initial distribution ν.

108. Let X_1 and X_2 be two independent random variables with $X_1\sim$ binomial $\left(m,\frac{1}{2}\right)$ and $X_2\sim$ binomial $\left(n,\frac{1}{2}\right)$, $m\neq n$. Which of the following are always true?

1. $2X_1+3X_2\sim$ binomial $\left(2m+3n,\frac{1}{2}\right)$.
2. $X_2-X_1+m\sim$ binomial $\left(m+n,\frac{1}{2}\right)$.
3. Conditional distribution of X_2 given (X_1+X_2) is hypergeometric.
4. Distribution of X_1-X_2 is symmetric about 0.

109. Let $X_1, X_2,\ldots,X_n$, $(n\geq 3)$ be a random sample from uniform $(\theta-5, \theta-3)$. Let $X_{(1)}$ and $X_{(n)}$ denote the smallest and largest of the sample values. Then which of the following are always true?

1. $(X_{(1)}, X_{(n)})$ is complete sufficient for θ.
2. $X_1+X_2-2X_3$ is an ancillary statistic.
3. $X_{(n)}+3$ is unbiased for θ.
4. $X_{(1)}+5$ is consistent for θ.

110. Let $X\sim$ Poisson (θ), the prior distribution of θ be exponential with median $\log_e 2$ and the loss function be squared error. Then which of the following are necessarily true?

1. The posterior distribution is Gamma.
2. The prior is a conjugate prior.
3. The posterior mean is $\dfrac{(x+1)\log_e 2}{1+\log_e 2}$.
4. The Bayes estimator of θ is $\dfrac{X+1}{2}$.

111. Let X_1, X_2 and X_3 be independent with $X_1 \sim N(1, 1)$, $X_2 \sim N(-1, 1)$ and $X_3 \sim N(0, 1)$. Let

$$q_1 = \frac{X_1^2 + X_2^2 + 2X_3^2 + 2X_1X_2}{2},$$

$$q_2 = \frac{X_1^2 + X_2^2 - 2X_1X_2}{2}.$$

Then which of the following statements are always true?

1. q_1 has a central chi-square distribution.
2. q_2 has a central chi-square distribution.
3. $q_1 + q_2$ has a central chi-square distribution.
4. q_1 and q_2 are independent.

112. The lifetimes (measured in hours) of a batch of lithium batteries are independent and identically distributed with density $f(t) = \lambda^2 te^{-\lambda t}$, $t \geq 0$, $\lambda > 0$. One battery from this batch is put to test and is observed to fail after 3 hours, while another battery is observed for 2 hours with no failure. Which of the following statements are necessarily correct?

1. The censoring involved here is of type II.
2. The likelihood for λ is proportional to $\lambda^2(1+3\lambda)e^{-5\lambda}$.
3. The maximum likelihood estimate of λ is 2 .
4. The maximum likelihood estimate of $\frac{1}{\lambda}$ is 2.

113. In the linear regression fit involving independent and identically distributed pairs of observations of a response variable (Y) and an explanatory variable (X), indicate which of the following statements are necessarily correct?

1. The residuals of the regression of Y on X have the same variance.
2. The multiple R^2 computed from the regression fit of Y on X and X^2 cannot be smaller than that computed from the regression fit of Y on X alone.
3. The adjusted R^2 computed from the regression fit of Y on X and X^2 cannot be smaller than that computed from the regression fit of Y on X alone.
4. The variance of the estimated regression coefficient of X computed from the regression of Y on X and X^2 is larger than that computed from the regression of Y on X alone.

114. Consider a linear model $E(\underset{\sim}{Y}) = X\underset{\sim}{\beta}$ where $\underset{\sim}{Y}$ is the vector of n observations and $\underset{\sim}{\beta}$ is the vector of n parameters. The $(i,j)^{th}$ element of the design matrix X is of the form: $1 + ij + i^2j^2$, $1 \leq i, \ j \leq$ n . Then $\underset{\sim}{\beta}$ is estimable for

1. n = 20.
2. n = 3.
3. n = 10.
4. n = 50.

115. For which of the following set of values will a Balanced Incomplete Block design with parameters v, b, r, k, λ not exist?

1. v = 11, b = 22, r = 6, k = 3, λ = 1.
2. v = 21, b = 4, r = 4, k = 21, λ = 4.
3. v = 7, b = 7, r = 4, k = 4, λ = 2.
4. v = 7, b = 7, r = 3, k = 3, λ = 1.

116. Consider a 2^3 factorial design laid out in 2 blocks, each of size 4, as follows

Block 1: | 1 a b c | Block 2: | ab ac bc abc |

Here the treatment combinations are written in Yates' notation. Then which of the following are always true?

1. Main effect A is confounded.
2. Main effect B is unconfounded.
3. Interactions AB, BC, AC are all unconfounded.
4. Interaction ABC is confounded.

117. Let $N(\underset{\sim}{\mu}, \Sigma)$ denote the multivariate normal distribution with mean vector $\underset{\sim}{\mu}$ and dispersion matrix Σ. Which of the following are correct?

1. The Bayes classifier between $N(\underset{\sim}{\mu}_1, \Sigma_1)$ and $N(\underset{\sim}{\mu}_2, \Sigma_2)$ consists of comparing a linear function of the classifiable vector with a threshold.
2. The misclassification probability of the Bayes classifier between the populations $N(\underset{\sim}{\mu}_1, \Sigma)$ and $N(\underset{\sim}{\mu}_2, \Sigma)$ is more than that of the Bayes classifier between the populations $N(2\underset{\sim}{\mu}_1, 2\Sigma)$ and $N(2\underset{\sim}{\mu}_2, 2\Sigma)$.
3. When $\underset{\sim}{\mu}_1$ and $\underset{\sim}{\mu}_2$ are unknown, the maximum likelihood classifier between $N(\underset{\sim}{\mu}_1, I)$ and $N(\underset{\sim}{\mu}_2, I)$, based on sampled data from the two populations, depends on the data only through the respective sample means.
4. The Bayes classifier between $N(1,0)$ and $N(10,4)$ would classify the data point 2 as belonging to the first population.

118. To study the relationship between smoking and lung cancer, 50 adult individuals are randomly selected from the population. For these 50 individuals, the responses to the questions (i) "Do you smoke?" (ii) "Do you have lung cancer?" lead to the following 2×2 table.

Smoking habit	Lung Cancer: Yes	Lung Cancer: No
Yes	15	5
No	10	20

Then which of the following statements are correct at 5% level of significance? (Note: 95^{th} percentile of the chi-square distribution with 1 d.f. is 3.841).

1. Lung cancer and smoking habits are significantly related.
2. Lung cancer and smoking habits are independent.
3. The proportions of individuals with lung cancer are significantly different in the smoking and non-smoking populations.
4. The proportions of smokers in the populations of individuals with or without lung cancer are significantly different.

119. Let X be a discrete random variable with probability mass function $p(x)$, $x \in \{-1,0,1,2,3\}$. The hypotheses to be tested are

$$H_0 : p(x) = p_0(x) \text{ versus } H_1 : p(x) = p_1(x),$$

where $p_0(x)$ and $p_1(x)$ are as given below.

x	−1	0	1	2	3
$p_0(x)$	0.01	0.02	0.05	0.32	0.60
$p_1(x)$	0.04	0.08	0.25	0.03	0.60

Then

1. {1} is the critical region of an MP test at level $\alpha = 0.05$.
2. {−1, 0} is the critical region of an MP test at level $\alpha = 0.03$.
3. {−1, 1} is the critical region of an MP test at level $\alpha = 0.06$.
4. {−1, 0, 1} is the critical region of an MP test at level $\alpha = 0.08$.

120. A linear programming problem

$$\max_{\underset{\sim}{x}} \underset{\sim}{c}'\underset{\sim}{x}, \quad \text{subject to } A\underset{\sim}{x} \le \underset{\sim}{b}, \quad \underset{\sim}{x} \ge \underset{\sim}{0},$$

attains the optimum value at two distinct feasible solutions. Which of the following must be true?

1. There must be infinitely many feasible solutions where the optimal value is attained.
2. All feasible solutions must be optimal solutions.
3. The dual problem has unbounded feasible region.
4. Rank of $\begin{pmatrix} \underset{\sim}{c}' \\ A \end{pmatrix}$ = Rank of A.

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MATHEMATICAL SCIENCES

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