Ordinary Differential Equations:

CENTER FOR MATHEMATICS AND STATISTICS, TRIVANDRUM

1. Consider inital value problem (IVP)

$$\frac{dy}{dx} = x^2|y|, \quad y(0) = 0, \quad x \in (-\infty, \infty).$$

Let $R : \{(x, y); |x| \leq a, |y| \leq b, a > 0, b > 0\}$ be a rectangle. Then

(a) $\frac{\partial f}{\partial y}$ fails to exists at many points in the rectangle R, where $f(x, y) = x^2|y|$.

(b) There exists more than one solution.

(c) $f(x, y) = x^2|y|$ is Lipschitz continuous in the rectangle R.

(d) The IVP has unique solution in a neighborhood of 0.

2. Consider the differential equation

$$(P) : \quad \frac{dy(x)}{dx} = |y(x)|, \quad x \in (-\infty, \infty).$$

Then

(a) If $y(0) = y_0$, for any $y_0 > 0$, there is unique solution.

(b) every solution of (P) is of the form $y = ce^x, c \geq 0$.

(c) If $y(0) = y_0$, for some $y_0 < 0$, there is no solution.

(d) there exist a solution $y(x)$ such that $y(x) < 0$, for all x.

3. Consider the initial value problem (IVP)

$$y'(t) = e^{-t^2} y(t), \quad y(e) = 0.5,$$

. Then, the IVP

(a) has infinitely many solutions

(b) has a solution which is not defined at some $t \in \mathbb{R}$

(c) has a unique solution in $\mathbb{R}$

(d) has no solution

4. Let $y : \mathbb{R} \to \mathbb{R}$ be differentiable and satisfy the ODE:

$$\frac{dy}{dx} = f(y), \quad x \in \mathbb{R}$$
$$y(0) = y(1) = 0$$

where $f : \mathbb{R} \to \mathbb{R}$ is a Lipschitz continuous function. Then

(a) $y(x) = 0$ if and only if $x \in \{0, 1\}$

(b) y is bounded

(c) y is strictly increasing

(d) $\frac{dy}{dx}$ is unbounded

5. Consider the solution of the ordinary differential equation

$$y'(t) = -y^3 + y^2 + 2y, \quad \text{subject to} \quad y(0) = y_0 \in (0, 2).$$

Then $\lim_{t \to \infty} y(t)$ belongs to

(a) $\{-1, 0\}$

(b) $\{-1, 2\}$

(c) $\{0, 2\}$

(d) $\{0, \infty\}$

6. If the solution to

$$\begin{cases} \dfrac{dy}{dx} = y^2 + x^2, & x > 0 \\ y(0) = 2 \end{cases}$$

exists in the interval $[0, L_0)$ and the maximal interval of exixtence of

$$\begin{cases} \dfrac{dz}{dx} = z^2, & x > 0 \\ z(0) = 1 \end{cases}$$

is $[0, L_1)$, then which of the following statements are correct?

(a) $L_1 = 1, \quad L_0 > 1$

(b) $L_1 = 1, \quad L_0 \leq 1$

(c) $L_1 < 2, \quad L_0 \leq 1$

(d) $L_1 > 2, \quad L_0 < 1$

7. Ler $f : \mathbb{R} \to \mathbb{R}$ be a Lipschitz function such that $f(x) = 0$ if and only if $x = \pm n^2$ where $n \in N$. Consider the initial value problem:

$$y'(t) = f(y(t)), \quad y(0) = y_0.$$

Then which of the following are true?

(a) y is a monotone function for all $y_0 \in \mathbb{R}$

(b) For any $y_0 \in \mathbb{R}$, there exists $M_{y_0} > 0$ such that $|y(t)| \leq M_{y_0}$ for all $t \in \mathbb{R}$

(c) there exists $y_0 \in \mathbb{R}$, such that the corresponding y is unbounded

(d) $\sup\limits_{t,s\in\mathbb{R}} |y(t) - y(s)| = 2n + 1$ if $y_0 \in (n^2, (n+1)^2), \quad n \geq 1$

8. Consider the initial value problem $\dfrac{dy}{dx} = x^2 + y^2, \quad y(0) = 1; \quad 0 \leq x \leq 1$. Then which of the following statements are true?

(a) There exists a unique solution in $[0, \frac{\pi}{4}]$

(b) Every solution is bounded in $[0, \frac{\pi}{4}]$

(c) The solution exhibits a singularity at some point in $[0, 1]$

(d) The solution becomes unbounded in some subinterval of $[\frac{\pi}{4}, 1]$

9. Consider the initial value problem

$$y'(t) = f(y(t)), \quad y(0) = a \in \mathbb{R}, \quad \text{where} \quad f : \mathbb{R} \to \mathbb{R}.$$

Which of the following statements are necessarily true ?

(a) There exists a continuous function $f : \mathbb{R} \to \mathbb{R}$ and $a \in \mathbb{R}$ such that the above problem does not have a solution in any neighbourhood of 0.

(b) The problem has a unique solution for every $a \in \mathbb{R}$ when f is Lipschitz continuous.

(c) When f is twice continuously differentiable, the maximal interval of existence for the above initial value problem is $\mathbb{R}$.

(d) The maximal interval of existence for the problem is $\mathbb{R}$ when f is bounded and continuously differentiable.

10. Let y be a solution of $y' = e^{-y^2} - 1$ on $[0, 1]$ which satisfies $y(0) = 0$. Then

(a) y is strictly increasing for $x > 0$

(b) y is strictly decreasing for $x > 0$

(c) $y(x) \neq 0$ for some $x > 0$.

(d) y has infinitely many zeros.